INVARIANTS OF ANTISYMMETRIC TENSORS

By

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This dissertation investigates invariants of antisymmetric tensors. It was motivated by problems that occurred in interpreting invariants that arose in certain projective geometry contexts (e.g., rigidity, convex polytopes). The goal was to contribute to the techniques for unraveling the geometry encoded in these invariants.

The principal tool was a simple yet previously unexploited observation that gives a realization of linear invariant functions of extensors as bracket polynomials where the extensor arguments are conspicuous. We call this realization dotted brackets. These readily provide examples of invariants, some of which are interpreted.

We represent dotted bracket expressions with tableaux and establish identities in tableaux. These translate to bracket polynomial identities which preserve the dotted bracket condition.

Tableaux identities are employed in a straightening algorithm. The straightening algorithm gives representations of tableaux expressions in a standard basis. This allows comparison of tableaux expressions.

Other methods of analyzing invariants are examined, most notably viewing the quadratic space of two–extensors determined by the Plucker relation on $\Lambda^2(R^4)$. The effectiveness of this perspective is demonstrated on a line geometry problem.

Throughout this dissertation there are projective geometry observations some of which are technical lemmas used to establish other results and some which are of interest for their own sake and whose verification lends itself to the methods developed here.

CHAPTER 1

INTRODUCTION

A Motivating Example.

The figure below is a bar and joint framework in the plane. It represents a set of rigid bars connected at flexible joints a, b, c, d, e and f which are points in rank–three projective space.

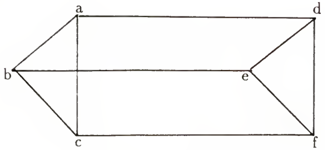

Figure 1.1

Of interest to structural engineers is the question: Is a framework rigid, i.e. does it allow motion (perhaps infinitesimal motion) at a joint? In 1983 White and Whiteley [24] discovered an expression called the pure condition that resolves this question. The pure condition is a determinant polynomial that is derived by an explicit combinatorial algorithm. A framework has an infinitesimal motion if and only if the pure condition is zero. The pure condition for this example is:

$$[edf][acd][cbf][bae] + [edf][abd][caf][bce]$$

Not so immediate from the pure condition is the relationship between the components of the framework, the points and lines (and planes, etc. in higher dimensional structures), that give a zero value to the pure condition.

The pure condition in the example is an invariant as a function of the underlying vectors and can be realized as a function some of whose arguments are lines of the configuration. Such arguments are extensors, the objects of the exterior algebra on the underlying vector space. Thus the pure condition is an invariant of extensors. We will return to this example.

Invariants like the pure condition arise in such stolid applications as rigidity of frameworks and satellite geodesy [7] and in more esoteric applications like oriented matroids and computational theorem proving [19]. It is this interest that motivated this research.

A Brief Historical Sketch

Invariant theory has experienced a series of revivals over the past 100 years. Interest focused on the subject with Felix Klein's edict in his Erlanger Programm, which stated that geometry is the study of properties that are invariant under the action of some transformation group.

The first major installment in the theory was the discovery of the symbolic method. Invariants could be realized as synthetic expressions, tableaux containing generic variables [18]. However the process of unravelling the geometry encoded in these symbolic expressions was excruciating. Indeed, mathematicians such as R. Weitzenbock and A. Cayley carried symbolic expressions around for years attempting to interpret them.

The first fundamental theorem of invariant theory established that vector invariants can be realized as polynomials in determinants. The second fundamental theorem of invariant theory spells out the determinant identities. This led to the straightening algorithm and to a canonical system for realizing invariants. The standard basis theorem establishes standard Young tableaux as a vector space basis for

vector invariant functions [15]. These breakthroughs resolved many of the prevailing problems in invariant theory of that day. An enigma remained though. Determinant polynomials allowed for some interpretation of invariants but many were too elaborate to reveal the geometry they encoded. Also the problem of how to employ invariants to state a geometry theorem remained.

Paralleling the development of invariant theory was the unfolding of the Grassmann calculus. In its context Grassmann's development was a most amazing discovery. It also has an interesting and tragic history. Theorems like the following are contained in Euclid:

If D is a point on side BC of triangle ABC, then $\dfrac{BD}{BC} = \dfrac{ABD}{ABC}$.

This suggests a product that joins two points to form a line or joins three points to form a plane. This product is exactly what Grassmann invented. It is known today as the wedge product of the exterior algebra, $\Lambda(V)$ of vector space V (or more currently as the join of the Cayley Algebra). What is amazing about Grassmann's invention is that the basic concepts of linear algebra like independence, basis, etc., were not formalized at this time. Many of these notions are contained either implicitly or explicitly in Grassmann's work. A case has been made that Grassmann invented linear algebra [11]. The tragedy is that Grassmann's endeavor went largely unappreciated by his contemporaries (with a few notable exceptions, e.g. Peano [8]). Many twentieth century geometers have emphasized the need for Grassmann to get his due.

Grassmann's calculus gained the appreciation it deserved with the advent of linear algebra and the exterior algebra. It saw steady application to geometry. Forder's text [12] is a litany of theorems stated in the Grassmann calculus. Since invariant theory is geometry, Forder's book is strewn with invariants.

In 1976 Doubilet, Rota and Stein [10] gave a novel perspective of Grassmann's system as a Cayley space, a vector space endowed with a bracket, i.e. a nondegenerate

alternating form. This curiously overlooked appendage extended the richness of the bracket identities or syzygies to the Grassmann calculus and opened the path between geometry and determinant polynomials. Doubilet, Rota and Stein called their system the Cayley algebra. Whiteley proved that all theorems of projective geometry can be realized as identities in the Cayley algebra [30].

The 1980s have seen an intensified interest in invariant theory. Besides classical geometry problems, techniques in such considerations as structural rigidity, convex polytopes, and more recently, oriented matroids [5] generate bracket polynomials. These beg for invariant theoretic analysis. The Cayley factorization algorithm [21, 25] is a significant advance in decoding the geometry of bracket polynomials. There have been other interesting results related to this dissertation. In 1983 White discovered an invariant called the superbracket [28] which is zero precisely when there is a dependency among six two–extensors of rank–four projective space. Huang [16] has given a description of invariants of two–extensors in rank–four projective space. In a much broader setting Grosshans, Rota and Stein [13] have refined the symbolic method and extended the standard basis theorem. It is in this setting that this research lies.

The Cayley Algebra

Let V be a vector space of dimension n over a field K. The exterior algebra of V is $\wedge(V)$ [11]. We adopt the current convention of representing the exterior (wedge) product with $\vee$ rather than $\wedge$ and refer to the operation as **join**. The join is associative and distributive over addition and is antisymmetric.

Let $a_1, \ldots, a_k \in V$. Then $A = a_1 \vee a_2 \vee \ldots \vee a_k = a_1 a_2 \ldots a_k$ is a **step k extensor**. If $B = b_1 \ldots b_j$ is a step–j extensor, then $A \vee B = a_1 \ldots a_k b_1 \ldots b_j$ is a

step-$(j+k)$ extensor. $A \vee B \neq 0$ if and only if $a_1, \ldots, a_k, b_1, \ldots, b_j$ are linearly independent.

The objects of the Cayley algebra are tensors. An extensor of step-k is a tensor that can be realized as a join of k vectors. Extensors are sometimes called decomposable tensors, versus non-decomposable tensors which can be realized only as a sum of extensors.

For n vectors $a_1 a_2, \ldots, a_n$, $[a_1 \ldots a_n]$ is the **bracket** of $a_1, \ldots, a_n$. The standard model of the bracket is the determinant. When coordinates are specified, the bracket corresponds to the determinant of the matrix whose columns are the vectors. An n-extensor $a_1 \ldots a_n$ is $[a_1 \ldots a_n]$ up to a fixed non-zero multiple.

Let $C = c_1 \ldots c_k$ and $B = b_1 \ldots b_k$ be two extensors of step-k. Extensors B and C are equivalent, denoted $B \sim C$, if and only if $[c_1 \ldots c_k x_1 \ldots x_{n-k}] = [b_1 \ldots b_k x_1 \ldots x_{n-k}]$ where $x_1, \ldots, x_{n-k}$ are indeterminant vectors.

Now we define a second operation which corresponds to Grassmann's regressive product. We use the symbol $\wedge$ and call the operation the **meet**. Let $A = a_1 \ldots a_k$ and $B = b_1 \ldots b_j$ where $k + j \geq n$. Then define

$$A \wedge B = \sum \operatorname{sign}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)} \cdots, a_{\sigma(n-j)}, b_1, \ldots, b_j] a_{\sigma(n-j+1)} \cdots a_{\sigma(k)}.$$

The sum is over all permutations σ of $\{1, 2, \ldots, k\}$ such that $\sigma(1) < \ldots < \sigma(n-j)$ and $\sigma(n-j+1) < \ldots < \sigma(k)$. Such permutations are called **shuffles** of the $(n-j, k-(n-j))$ split of A. An alternate notation for signed sums of shuffles over splits is the Scottish notation used informally by Turnbull [23]. In this notation a dot is placed over the shuffled vectors with the summation and sign(σ) implicit. The brackets effect the partition of the split. So

$$A \wedge B = [\dot{a}_1 \dot{a}_2 \ldots \dot{a}_{n-j} b_1 \ldots b_j] \dot{a}_{n-j+1} \ldots \dot{a}_k.$$

We will make extensive use of this notation and on occasions sum over more than one shuffle of disjoint sets of vectors, in which cases we use other symbols (triangles, squares, etc.) over the intended vectors.

All of the invariants examined in this dissertation are in the context of projective geometry. We employ the standard homogeneous coordinatization of points (and subsequently lines, planes, etc.). A point in n-dimensional affine space $(x_1, x_2, \ldots, x_n)$ is associated with the point $(cx_1, cx_2, \ldots, cx_n, a)$ where $a \neq 0$. In this coordinatization $(y_1, \ldots, y_{n+1}) = (cy_1, \ldots, cy_{n+1})$. The homogeneous coordinatizations with last coordinate zero are the "points at infinity" of projective geometry. Homogeneous coordinates give a vector space coordinatization of projective space. We refer to the space associated with affine n-dimensional space as projective rank-$n+1$ space.

EXAMPLE 1.1: In rank-four projective space,

$$\begin{aligned} a_1a_2a_3 \wedge b_1b_2b_3 &= [\dot{a}_1b_1b_2b_3]\dot{a}_2\dot{a}_3 \\ &= [a_1b_1b_2b_3]a_2a_3 - [a_2b_1b_2b_3]a_1a_3 + [a_3b_1b_2b_3]a_1a_2 \end{aligned}$$

The step-two tensor in the expanded sum corresponds to the line of intersection of the planes determined by a_1, a_2, a_3 and b_1, b_2, b_3 respectively.

We remark that the extension of the exterior algebra to a Cayley space allows for a succinct definition of the meet operation [10]. Previously the best definitions of the Grassmann regressive product were in terms of duality of vector spaces.

We also note that when the component vectors of the extensors are independent, the join operation corresponds to the join in the subspace lattice of V. And when component vectors span V, the meet operation corresponds to the meet in the subspace lattice. This is the justification for the terminology that deviates from the familiar exterior algebra notation.

Besides a much cleaner definition of the regressive product, the bracket brings to the Cayley algebra its syzygies. These are the principal instrument in the straighten-

ing algorithm that establishes the standard basis theorem that in turn allows comparison of Cayley algebra expressions. The basic syzygy says that over an n dimensional vector space a split sum over $m > n$ vectors is zero [26].

EXAMPLE 1.2: In rank–four projective space,

$$
\begin{aligned}
[a\dot{b}\dot{c}\dot{d}][\dot{e}\dot{f}gh] &= [abcd][efgh] - [abce][dfgh] + [abcf][degh] \\
&\quad + [abde][cfgh] - [abdf][cegh] - [acde][bfgh] \\
&\quad + [acdf][begh] + [abef][cdgh] - [acef][bdgh] \\
&\quad + [adef][bcgh] \\
&= 0.
\end{aligned}
$$

These syzygies are a consequence of a basic multilinear algebra fact. If f is a multilinear function over an n dimensional vector space that is antisymmetric in $m > n$ of its vector arguments, then $f = 0$.

The syzygy says

$$
\begin{aligned}
&[\ldots \dot{a}_1 \dot{a}_2 \ldots] \ldots [\ldots \dot{a}_j \ldots \dot{a}_m] \\
&\quad = \sum_{\sigma} \operatorname{sign}(\sigma)[\ldots a_{\sigma 1} a_{\sigma 2} \ldots] \ldots [\ldots a_{\sigma j} \ldots a_{\sigma m}] = 0.
\end{aligned}
$$

where the sum is over the split–shuffles of the a_i and $m > n$, n the dimension of the underlying vector space. In typical applications like the straightening algorithm we use the consequence

$$
\begin{aligned}
&[\ldots a_1 a_2 \ldots] \ldots [\ldots a_j \ldots a_m \ldots] \\
&\quad = - \sum_{\sigma \neq I} \operatorname{sign}(\sigma)[\ldots a_{\sigma 1} a_{\sigma 2} \ldots] \ldots [\ldots a_{\sigma j} \ldots a_{\sigma m}].
\end{aligned}
$$

We substitute the summation for monomial terms. When employing the substitution, we designate the vectors in the monomial to be shuffled in the syzygy by boldface.

EXAMPLE 1.3: We illustrate the syzygy and a multiple dotting. The expression is an invariant of the two–extensors ab, cd, ef and gh. The equal expression that we realize has the component vectors (factors) of the extensor arguments together in brackets.

$$[ab\dot{c}\overset{\blacktriangle}{e}][\dot{d}\overset{\blacktriangle}{f}gh] = [abc\mathbf{e}][\mathbf{dfgh}] - [abcf][degh] - [abde][cfgh] + [abdf][cegh]$$

$$= ([abcd][efgh] + [abcf][degh] - [abcg][defh] + [abch][defg]$$

$$+ [abde][cfgh] + [abef][cdgh] - [abeg][cdfh] + [abeh][cdfg]$$

$$- [abdf][cegh] + [abdg][cefh] - [abdh][cefg]$$

$$+ [abeg][cdfh] - [abfh][cdeg] + [abgh][cdef])$$

$$- [abcf][degh] - [abde][cfgh] + [abdf][cegh].$$

Now doing the immediate cancellations, rearranging the terms and adding in and subtracting out the term $[abgh][cdef]$ in an appropriate place, we see that the expression above is

$$[abcd][efgh] + [abef][cdgh] - [abgh][cdef] + ([abgh][cdef]$$

$$- [abcg][defh] + [abdg][cefh] - [abeg][cdfh] + [abfg][cdeh])$$

$$+ ([abgh][cdef] + [abch][defg] - [abdh][cefg]$$

$$+ [abeh][cefg] - [abfh][cdeg]$$

$$= [abcd][efgh] + [abef][cdgh] - [abgh][cdef]$$

$$+ [abg\dot{h}][\dot{c}\dot{d}\dot{e}\dot{f}] + [ab\dot{g}h][\dot{c}\dot{d}\dot{e}\dot{f}]$$

$$= [abcd][efgh] + [abef][cdgh] - [abgh][cdef]$$

At this point this example is a moderately elaborate illustration of some definitions and application of identities. We will refer back to this example. We will see that it is an interesting invariant with a geometric interpretation. Moreover, it will indicate the economy of methods to be developed in this dissertation.

The following example illustrates the way geometry theorems are encoded in Cayley algebra expressions.

EXAMPLE 1.4: Pappus's Theorem (projective rank–three).

If points a, b and c are collinear and points a', b' and c' are collinear, then points $(ab' \wedge a'b)$, $(a'c \wedge ac')$ and $(bc' \wedge b'c)$ are collinear.

PROOF:

$$
\begin{aligned}
&(ab' \wedge a'b) \vee (a'c \wedge ac') \vee (bc' \wedge b'c) \\
&= ([aa'b]b' - [a'bb']a) \vee ([acc']a' - [aa'c']c) \vee ([bb'c]c' - [b'cc']b) \\
&= -[aa'b][acc'][bb'c][a'b'c'] - [aa'b][acc'][b'cc'][a'bb'] \\
&- [aa'b][aa'c'][bb'c][b'cc'] + [aa'b][aa'c'][b'cc'][bb'c] \\
&- [a'bb'][acc'][bb'c][aa'c'] + [a'bb'][acc'][b'cc'][aa'b] \\
&+ [a'bb'][aa'c'][bb'c][acc'] + [a'bb'][aa'c'][b'cc'][abc] \\
\\
&= [a'bb'][aa'c'][b'cc'][abc] - [aa'b][acc'][bb'c][a'b'c'] \\
&= 0 \text{ when } a, b \text{ and } c \text{ are collinear and } a', b' \text{ and } c' \text{ are collinear.}
\end{aligned}
$$

■

To take a determinant polynomial and unravel the geometry that it encodes. one must decompose it in the Cayley Algebra. We illustrate this with the following example.

EXAMPLE 1.5: Recall the pure condition for the bar and joint framework presented earlier in this chapter.

$$
\begin{aligned}
&[edf][acd][cbf][bae] + [edf][abd][caf][bce] \\
&= [abc][def](ad \vee be \vee cf).
\end{aligned}
$$

This expression is zero, and thus the framework has an infinitesimal motion, exactly when lines ad, be, and cf are concurrent or triangles abc or def are degenerate.

White's Cayley factorization algorithm [25] decomposes linear invariants in the Cayley algebra. This substantial result has had a symbiotic relationship with this research [21].

Spaces of Tensors

We will have occasion to be interested in the subspaces of tensors that are spanned by certain sets of tensors. We will deal almost exclusively with spaces of lines (two–extensors) in rank–four projective space. These spaces are well known and categorized [6]. There are several interesting spaces of lines.

If $A, B, \ldots, D$ are extensors of the same step, we denote the span by $\langle A, B, \ldots, D\rangle$. In Chapter 4 we define a bilinear form on two–tensors. The form being zero on a pair of two–tensors gives a notion of orthogonality. We denote the space orthogonal to $\langle A, B, \ldots, D\rangle$ by $\langle A, B, \ldots, D\rangle^{\perp}$.

EXAMPLE 1.6: If l_1, l_2 and l_3 are three mutually skew lines in projective rank–four space, then the family of lines of $\langle l_1, l_2, l_3\rangle$ is called a regulus. The family of lines that meet all the lines of $\langle l_1, l_2, l_3\rangle$ is the regulus conjugate to $\langle l_1, l_2, l_3\rangle$. We will denote it $\langle l_1, l_2, l_3\rangle^{\perp}$. A regulus and a conjugate regulus give a double ruling of a quadric surface.

CHAPTER 2

EXTENSOR INVARIANTS AS DOTTED BRACKETS

Let V be a vector space of dimension n over a field K. Let $X = X_{n \times m} = (X_{ij})$ denote an $n \times m$ matrix of commuting indeterminants. Let $F[X_{n\times m}] = F[X_{ij} : 1 \leq i \leq n,\ 1 \leq j \leq n]$. Then $f(X_{n\times m}) \in F[X_{n\times m}]$ is a vector invariant of the projective linear group, $\mathrm{PLG}(n, K)$, if $f(AX_{n\times m}) = (\det A)^k f(X_{n\times m})$ for all $A \in \mathrm{PLG}(n, K)$ where k is a positive integer. The first fundamental theorem of vector invariants states that the ring of invariants of $\mathrm{PLG}(n, K)$in $F[X_{n\times m}]$ is generated as a K-algebra by the $n \times n$ minors of $X_{n\times m}$. That is, these invariants can be realized as polynomials in brackets that are homogeneous both in bracket degree and in the occurrence of vectors as entries of brackets in the terms of the polynomials.

We will investigate multilinear invariant functions of skew-symmetric tensors. If $f(A_1, A_2, \dots, A_j, \dots, A_l)$ is a function of step k_i tensors A_i, $1 \leq i \leq l$, and $A_j \sim B$, then $f(A_1, A_2, \dots, A_j, \dots, A_l) = f(A_1, A_2, \dots, B, \dots, A_l)$. This is the requirement that the function of tensors be well defined. If T is a transformation in the projective linear group, then as a transformation of the underlying vector space, T effects a transformation on tensors. If $T \in \mathrm{PLG}(n, K)$ and $A = a_1a_2\dots a_k$, then $TA = Ta_1Ta_2\dots Ta_k$. By invariant functions we mean functions which are invariant as functions of tensors under actions of the projective linear group on the underlying vector space.

$f(A_1, A_2, \dots, A_l)$ is a **multilinear** function of the step k_i tensors, A_i if:

a) $f(A_1, A_2, \dots, A_j + B_j, \dots, A_l)$

$$= f(A_1, A_2, \dots, A_j, \dots, A_l) + f(A_1, A_2, \dots, B_j, \dots, A_l)$$

b) $f(A_1, A_2, \dots, \alpha A_j, \dots, A_l) = \alpha f(A_1, A_2, \dots, A_j, \dots, A_l)$, $\alpha \in K$.

We specialize our observations to functions of extensors, i.e. decomposable tensors. Most of the observations that we make about multilinear functions of extensors will readily extend to functions of tensors by multilinearity.

Let $A_i = a_{i,1}a_{i,2}\dots a_{i,k_i}, 1 \leq i \leq j$, be extensors of step k_i. A function $f(A_1, A_2, \dots, A_j)$ induces a function of the vectors $a_{1,1}, a_{1,2}, \dots a_{j,k_j}$, which are factors of the extensors in the Cayley algebra.

THEOREM 2.1. *A multilinear function of extensors, $f(a_1a_2\dots a_{k_1}, \dots, z_1\dots z_{k_l})$, is multilinear as a function $g(a_1, a_2, \dots, z_1, \dots, z_{k_l})$ of the vector factors of the extensor arguments of f.*

PROOF:

$$
\begin{aligned}
&g(a_1, a_2, \dots, x_j' + x_j'', \dots, z_{k_l}) \\
&= f(a_1a_2\dots a_{k_1}, \dots, x_1\dots x_{j-1}(x_j' + x_j'')x_{j+1}\dots x_{k_p}, \dots z_1\dots z_{k_l}) \\
&= f(a_1a_2\dots a_{k_1}, \dots, (x_1\dots x_{j-1}x_j'x_{j+1}\dots x_{k_p}) + \\
&\qquad\quad (x_1\dots x_{j-1}x_j''x_{j+1}\dots x_{k_p}), \dots, z_1\dots z_{k_l}) \\
&= f(a_1a_2\dots a_{k_1}, \dots, x_1\dots x_{j-1}x_j'x_{j+1}\dots x_{k_p}, \dots z_1\dots z_{k_l}) + \\
&\qquad\quad f(a_1a_2\dots a_{k_1}, \dots, x_1\dots x_{j-1}x_j''x_{j+1}\dots x_{k_p}, \dots z_1\dots z_{k_l}) \\
&= g(a_1, \dots, x_j', \dots, z_{k_l}) + g(a_1, \dots, x_j'', \dots, z_{k_l}).
\end{aligned}
$$

Similarly $g(a_1, \dots, \alpha x_j, \dots, z_{k_l}) = \alpha g(a_1, \dots, x_j, \dots, z_{k_l})$ and thus g is multilinear.

■

If $f(A_1, A_2, \dots, A_j)$ is a multilinear invariant function of extensors, then as a function of the vector factors of the A_i, f has a realization as a bracket polynomial that is multilinear in the vector factors.

EXAMPLE 2.2: In rank-4 Real Projective Space, $f(ab, cd, ef, gh){=}g(a, b, c, d, e, f, g, h)$ $= [abcd][efgh] + [abef][cdgh] + [abgh][cdef]$ is a multilinear invariant function of the step-2 extensors ab, cd, ef, and gh.

The first fundamental theorem of vector invariants guarantees that a multilinear function of extensors invariant over the projective linear group has a realization as a bracket polynomial that is multilinear in the vector factors of the extensors. It is evident that not every such bracket polynomial is a multilinear invariant function of the extensor arguments which we obtain by a partition of the vector arguments of the function, joining the vectors of the parts to form extensors. The problem is to recognize those bracket polynomials which are multilinear invariant functions for a prescribed joining of vectors into extensors. We will establish a realization for multilinear invariant functions in terms of the component vectors of extensors. Bracket identities can conceal this realization, but any invariant function of extensors will be equal to such an expression up to the syzygies.

By skew symmetry the extensor $a_1a_2\dots a_k = \text{sign}(\sigma)\ a_{\sigma 1}a_{\sigma 2}\dots a_{\sigma k}, \sigma \in S_k$. If $f(a_1\dots a_{k_1}, b_1\dots b_{k_2}, \dots, z_1\dots z_{k_l})$ is a multilinear invariant function of the step-k_i extensors $a_1\dots a_{k_1}, \dots, z_1\dots z_{k_l}$, then by linearity

$$f(a_1a_2\dots a_{k_1}, b_1b_2\dots b_{k_2}, \dots, z_1\dots z_{k_l})$$
$$= \text{sign}(\sigma)\ f(a_{\sigma 1}a_{\sigma 2}\dots a_{\sigma k_1}, b_1b_2\dots b_{k_2}, z_1\dots z_{k_l}).$$

We can extend this to make the following observation.

LEMMA 2.3. *If* $f(a_1a_2\dots a_{k_1}, b_1b_2\dots b_{k_2}, \dots, z_1\dots z_{k_l})$ *is a multilinear invariant function of the step-*k_i *extensors* $a_1a_2\dots a_{k_1}, b_1b_2\dots b_{k_2}, \dots, z_1\dots z_{k_l}$, *then*

$$\sum_{\sigma_a\in s_{k_1}, \sigma_b\in s_{k_2}, \dots, \sigma_z\in s_{k_l}} \text{sign}(\sigma_a)\text{sign}(\sigma_b)\dots\text{sign}(\sigma_z)$$
$$f(a_{\sigma_{a}1}a_{\sigma_{a}2}\dots a_{\sigma_a k_1}, b_{\sigma_{b}1}b_{\sigma_{b}2}\dots b_{\sigma_b k_2}, \dots, z_{\sigma_{z}1}\dots z_{\sigma_z k_l})$$
$$=(k_1)!(k_2)!\dots(k_l)!\, f(a_1a_2\dots a_{k_1}, b_1b_2\dots b_{k_2}, \dots, z_1\dots z_{k_l}).$$

LEMMA 2.4. *Let*

$$f(a_1, a_2, \dots, a_k) = [a_1 a_2 \dots a_{p_1} b \dots c][a_{p_1+1}\, a_{p_1+2} \dots a_{p_1+p_2}\, d \dots e] \dots$$

$$\times [a_{p_1+p_2+\dots+p_{r-1}+1} a_{p_1+p_2+\dots+p_{r-1}+2} \dots a_k y \dots z],$$

where the a_i are vectors and $b, c, \dots z$ are fixed indeterminant vectors. The function f is a product of r $n \times n$ determinants among whose columns are the vectors $a_1, a_2, \dots, a_k$ occurring exactly once each with the brackets filled out with the indeterminant vectors $b, c, \dots, z$. We are assuming there are p_j occurrences of the a_i in the jth bracket of the product, $1 \le j \le r$. Then $\sum_{\sigma \in S_k}$ sign $(\sigma)\, f(a_{\sigma 1}, a_{\sigma 2}, \dots a_{\sigma k})$ is a dotted bracket expansion.

PROOF:

$$\sum_{\sigma \in S_k} sign(\sigma)\ f(a_{\sigma 1}, a_{\sigma 2}, \dots a_{\sigma k})$$
$$= \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, [a_{\sigma 1} a_{\sigma 2} \dots a_{\sigma p_1} b \dots c] \dots [a_{\sigma(p_1+p_2+\dots+p_{r-1}+1)} \dots a_{\sigma k} y \dots z].$$

Each permutation σ, together with the brackets, partitions the indices of the vectors $a_i, i = 1, 2, \dots k$. Let γ_σ be the permutation that orders the indices within each part: $\gamma_\sigma\sigma(1) < \gamma_\sigma\sigma(2) < \dots < \gamma_\sigma\sigma(p_1)$, $\gamma_\sigma\sigma(p_1+1) < \dots < \gamma_\sigma\sigma(p_1+p_2), \dots, \gamma_\sigma\sigma(p_1 + p_2 + \dots + p_{r-1} + 1) < \dots < \gamma_\sigma\sigma(k)$. The set $\{\gamma_\sigma\sigma(1),\ \gamma_\sigma\sigma(2), \dots, \gamma_\sigma\sigma(p_1)\}$ equals $\{\sigma(1),\ \sigma(2), \dots, \sigma(p_1)\}$, etc. Then from antisymmetry of the brackets,

$$f(a_1, a_2, \dots, a_k)$$
$$= \sum_{\sigma \in S_k} \operatorname{sign}(\gamma_\sigma\sigma)[a_{\gamma_\sigma\sigma 1} \dots a_{\gamma_\sigma\sigma p_1} b \dots c] \dots [a_{\gamma_\sigma\sigma(p_1+p_2+\dots+p_{r-1}+1)} \dots a_{\gamma_\sigma\sigma(k)} y \dots z].$$

In the $k!$ terms of the summation above there are $\prod_{i=1}^{r} (p_i)!$ occurrences of each term. Moreover if two permutations σ' and σ'' effect the same partition, i.e. the sets $\{\sigma' 1, \sigma' 2, \dots, \sigma' p\} = \{\sigma'' 1, \sigma'' 2, \dots, \sigma'' p\}, \dots, \{\sigma'(p_1+p_2+\dots+p_{r-1}+1), \dots, \sigma' k\} = \{\sigma''(p_1 + p_2 + \dots + p_{r-1} + 1), \dots, \sigma'' k\}$, then $\gamma_{\sigma'}\sigma' = \gamma_{\sigma''}\sigma''$. Thus the summands

indexed by σ' and σ'' have the same signature. The point is that there are no cancellations among the $\prod_{i=1}^{r}(p_i)!$ terms having the same bracket monomial. So

$$\sum_{\sigma\in S_k} \mathrm{sign}(\sigma)\, f(a_{\sigma 1}, a_{\sigma 2}, \dots, a_{\sigma k})$$

$$\prod_{i=1}^{r}(p_i)! \sum_{\delta} \mathrm{sign}(\delta)\, [a_{\delta 1}\dots a_{\delta p_1}\, b\dots c]\dots[a_{\delta(p_1+p_2+\dots+p_{r-1}+1)}\dots a_{\delta k} y\dots z]$$

where the δ are permutations of S_k of the form $\gamma_\sigma\sigma$; these are split shuffles, and they effect distinct partitions of $1, 2, \dots, k$ relative to the brackets. This summation is the expansion of the dotted bracket:

$$\prod_{i=1}^{r}(p_i)!\ [\dot{a}_1\dot{a}_2\dots\dot{a}_{p_1} b\dots c]\dots[\dot{a}_{p_1+p_2+\dots+p_{r-1}+1}\dots\dot{a}_k\, y\dots z]$$

.

■

THEOREM 2.5. *If $f(a_1a_2\dots a_{k_1}, b_1\dots b_{k_2}, \dots, z_1\dots z_{k_l})$ is a multilinear invariant function of the step-k_i extensors, $a_1a_2\dots a_{k_1}, \dots, z_1\dots z_{k_l}$, then f has a realization as a bracket polynomial that is a sum of dotted bracket expansions. The distinct split sums of each expansion are dottings of the vector factors of the distinct tensor arguments of f.*

PROOF: Let $f = g(a_1, a_2, \dots a_{k_1}, \dots, z_1, \dots, z_{k_l})$. The function g has a realization as a bracket polynomial homogeneous in the vector factors $a_1, a_2, \dots a_{k_1}, \dots, z_{k_l}$, $g = \sum[\ ][\ ]\dots[\ ]$. Applying Lemma 2.3,

$g =$

$$\frac{1}{(k_1)!(k_2)!\dots(k_l)!}\sum \mathrm{sign}(\sigma_a)\mathrm{sign}(\sigma_b)\dots\mathrm{sign}(\sigma_z) g(a_{\sigma_a 1}, \dots, a_{\sigma_a k_1}, b_{\sigma_b 1}, \dots, z_{\sigma_z k_l}).$$

Applying Lemma 2.4 inductively to the vectors, $a_1, a_2, \dots, z_{k_l}$ we realize g, hence f, as a sum of expanded dotted brackets with the dottings among the vector factors of the extensors. ■

The converse of Theorem 2.5 holds. The proof, however, at this point is laborious. Since this observation is an immediate consequence of subsequent results, we postpone the proof until Chapter 3. Then we will have established a compact form for multilinear invariants of extensors.

We present some examples of linear invariants together with interpretations of these under certain hypotheses. All of these initial examples are invariant functions of two-extensors.

EXAMPLE 2.6: If $f(ab, cd) = [abcd]$, then $f = 0$ if and only if lines ab and cd are coplanar, and thus intersect. This is evident since $[abcd]$ gives six times the volume of the tetrahedron with vertices a, b, c and d.

EXAMPLE 2.7: Let ab, cd and ef be distinct lines in rank-three projective space. Let $F(ab, cd, ef)$ be the linear invariant $[ab\overset{\bullet}{c}][\overset{\bullet}{d}ef] = [abc][def] - [abd][cef]$. Then $F = 0$ if and only if the point $(ab \wedge cd)$ lies on the line ef. Verifying this invariant is an easy Cayley algebra exercise. $F = (ab \wedge cd) \wedge ef = 0$ if and only if the lines ab, cd and ef are concurrent.

EXAMPLE 2.8: The superbracket, the linear invariant of six two-extensors in $\Lambda^2(R^4)$ which is zero precisely when there is a linear dependency among the extensors, has the following dotted bracket realization:

$$[ab\overset{\bullet}{g}\overset{\blacktriangle}{i}][cd\overset{\bullet}{h}\overset{\blacksquare}{k}][ef\overset{\blacktriangle}{j}\overset{\blacksquare}{l}] - [\overset{\circ}{a}\overset{\triangle}{c}gh][\overset{\circ}{b}\overset{\square}{e}ij][\overset{\triangle}{d}\overset{\square}{f}kl].$$

Actually this is not the original version of the superbracket presented by White [27]. This expression straightens to the original version by methods developed in this dissertation.

THEOREM 2.9. *Let ab, cd, and ef be distinct skew lines in rank-four projective space. Let $F(ab, cd, ef, gh)$ be the linear invariant $[ab\overset{\bullet}{c}\overset{\blacktriangle}{e}][\overset{\bullet}{d}\overset{\blacktriangle}{f}gh]$. Then $F = 0$ if and*

only if gh is in the five dimensional subspace of $\Lambda^2(R^4)$ (a line complex) spanned by cd, ef, and the regulus conjugate to the regulus of ab, cd, and ef.

PROOF: Let r_1, r_2, and r_3 be distinct lines of the regulus conjugate to the regulus of ab, cd and ef. In Chapter 4 we verify that three lines of a regulus and three lines of a conjugate regulus span $\Lambda^2(R^4)$. Using this fact we have

$$gh = \alpha_1 ab + \alpha_2 cd + \alpha_3 ef + \alpha_4 r_1 + \alpha_5 r_2 + \alpha_6 r_3.$$

From Example 1.3 $[ab\overset{\bullet}{c}\overset{\blacktriangle}{e}][\overset{\bullet}{d}\overset{\blacktriangle}{f}gh] = [abcd][efgh] + [abef][cdgh] - [abgh][cdef]$. Substituting $\alpha_1 ab + \alpha_2 cd + \alpha_3 ef + \alpha_4 r_1 + \alpha_5 r_2 + \alpha_6 r_3$ for gh in the right side of this equation we get

$$[ab\overset{\bullet}{c}\overset{\blacktriangle}{e}][\overset{\bullet}{d}\overset{\blacktriangle}{f}gh] = 2\alpha_1 [abcd][abef].$$

Therefore $[ab\overset{\bullet}{c}\overset{\blacktriangle}{e}][\overset{\bullet}{d}\overset{\blacktriangle}{f}gh] = 0$ if and only if $\alpha_1 = 0$, in which case gh is in the span of cd, ef, r_1, r_2, and r_3.

■

COROLLARY 2.10. *If ab, cd, and ef are mutually skewed, and*

$$[ab\overset{\bullet}{c}\overset{\blacktriangle}{e}][\overset{\bullet}{d}\overset{\blacktriangle}{f}gh] = 0$$

$$[\overset{\bullet}{a}\overset{\blacktriangle}{c}ef][\overset{\bullet}{b}\overset{\blacktriangle}{d}gh] = 0$$

and $$[\overset{\bullet}{a}cd\overset{\blacktriangle}{e}][\overset{\bullet}{b}\overset{\blacktriangle}{f}gh] = 0,$$

then gh is in the regulus conjugate to ab, cd, and ef.

CHAPTER 3

INVARIANTS AS TABLEAUX

We have established that linear invariant functions of extensors can be realized as dotted brackets. We will now achieve some notational convenience by equating dotted bracket expressions with tableaux. We will obtain identities for dotted brackets and then mimic some observations of classical invariant theory, in particular the work of Young that created a standard basis for the space of vector invariants [13, 18].

A tableau is an $n \times m$ array. The tableau entries will be lowercase Roman letters. We reserve uppercase letters to denote blocks of entries in rows of a tableau.

A realization of an invariant may be a sum of dotted brackets. We call a product of brackets with the dots in place, a **dotted bracket monomial**. Suppose we have a dotted bracket monomial that as a linear invariant is a function of the tensors $a_1a_2 \ldots a_{k_1}, b_1b_2 \ldots b_{k_2}, \ldots, d_1d_2 \ldots d_{k_j}$. We define this dotted bracket monomial to be equal to the following tableau:

1) Rows of the tableau correspond to brackets, the first row to the first bracket etc.

2) The entries of the rows will be the same letters as in the corresponding brackets but without subscripts.

3) There is a $\pm$ sign attached, that being sign(σ) where σ is the permutation that takes the bracket entries in the order they are given in the dotted bracket expression and orders them lexicographically, $a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}, \ldots$

We call these tableaux representing dotted brackets **compact tableaux** to distinguish them from ordinary tableaux which are associated with bracket monomials in the classical theory.

EXAMPLE 3.1.

$$[a_1a_2a_3\dot{c}_1][\overset{\blacktriangle}{b}_1\overset{\blacktriangle}{b}_2\dot{c}_2\dot{c}_3][\overset{\blacktriangle}{b}_3d_1d_2d_3] = -\begin{pmatrix} a & a & a & c \\ b & b & c & c \\ b & d & d & d \end{pmatrix}.$$

We have the negative sign since the odd permutation $\sigma = (b_1b_2b_3c_3c_2c_1)$ gives $\sigma\{a_1a_2a_3c_1b_1b_2c_2c_3b_3d_1d_2d_3\} = \{a_1a_2a_3b_1b_2b_3c_1c_2c_3d_1d_2d_3\}$.

Note that a particular tableau corresponds to different yet equal versions of a dotted expression. For example,

$$\begin{aligned} \begin{pmatrix} a & a & a & c \\ b & b & c & c \\ b & d & d & d \end{pmatrix} &= -[a_1a_2a_3\dot{c}_1][\overset{\blacktriangle}{b}_1\overset{\blacktriangle}{b}_2\dot{c}_2\dot{c}_3][\overset{\blacktriangle}{b}_3d_1d_3d_3] \\ &= [a_1a_2a_3\dot{c}_2][\overset{\blacktriangle}{b}_1\overset{\blacktriangle}{b}_2\dot{c}_1\dot{c}_3][\overset{\blacktriangle}{b}_3d_1d_2d_3] \\ &= [a_1a_2a_3\dot{c}_1][\overset{\blacktriangle}{b}_1\overset{\blacktriangle}{b}_3\dot{c}_2\dot{c}_3][\overset{\blacktriangle}{b}_2d_1d_2d_3], \end{aligned}$$

etc.

It will usually be our convention when associating a tableau with a dotted bracket to use the dotted bracket expression where the subscripts of each letter are in order across the bracket expression. So among all the choices we would use

$$\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} = -[a_1a_2\dot{b}_1\overset{\blacktriangle}{c}_1][\dot{b}_2\overset{\blacktriangle}{c}_2d_1d_2].$$

EXAMPLE 3.2. The superbracket (Example 2.8), the linear invariant function of six two–extensors in rank–four that is zero precisely when there is a linear dependency among its arguments, equals the following linear combination of compact tableaux.

$$\begin{pmatrix} a & a & d & e \\ b & b & d & f \\ c & c & e & f \end{pmatrix} - \begin{pmatrix} a & b & d & d \\ a & c & e & e \\ b & c & f & f \end{pmatrix}.$$

The dotted brackets impose an algebra on compact tableaux. We can realize linear invariant functions of tensors as sums of tableaux. We will establish identities in compact tableaux. First we make a preliminary observation.

Suppose T is a compact tableau in letters $a, b, c, d, e, \ldots$ with s occurrences of letter d; d corresponds to a step-k extensor in the equivalent dotted bracket expression. We show that T is equal to a sum of tableaux in *letters* $a, b, c, d_1, d_2, \ldots, d_s, e, \ldots$.

THEOREM 3.1. *Let T be a compact tableau on letters $a, b, c, d, e, \ldots$ with s occurrences of letter d, j in the first row, $k - j$ in the second row, etc. Then*

$$T = \begin{pmatrix} \ldots & X & \ldots & d & \ldots & d & \ldots & Y & \ldots \\ \ldots & Z & \ldots & d & \ldots & d & \ldots & W & \ldots \\ & & & & \vdots & & & & \\ \ldots & U & \ldots & d & \ldots & d & \ldots & V & \ldots \\ & & & & \vdots & & & & \end{pmatrix}$$

$$= \sum_\sigma T_\sigma = \sum_\sigma \begin{pmatrix} \ldots & X & \ldots & d_{\sigma 1} & \ldots & d_{\sigma j} & \ldots & Y & \ldots \\ \ldots & Z & \ldots & d_{\sigma(j+1)} & \ldots & d_{\sigma k} & \ldots & W & \ldots \\ & & & & \vdots & & & & \\ \ldots & U & \ldots & d_{\sigma(l+1)} & \ldots & d_{\sigma s} & \ldots & V & \ldots \\ & & & & \vdots & & & & \end{pmatrix},$$

where σ runs over all split-shuffles of the d_i, and where the T_σ are compact tableaux on letters $a, b, c, d_1, d_2, \ldots, d_s, e, f, \ldots$

PROOF:

$$T = \begin{pmatrix} \ldots & X & \ldots & d & \ldots & d & \ldots & Y & \ldots \\ \ldots & Z & \ldots & d & \ldots & d & \ldots & W & \ldots \\ & & & & \vdots & & & & \\ \ldots & U & \ldots & d & \ldots & d & \ldots & V & \ldots \\ & & & & \vdots & & & & \end{pmatrix}$$

$$= sign(T)[\overset{\bullet}{a}_1 \overset{\bullet}{a}_2 \ldots \overset{\blacktriangle}{d}_1 \overset{\blacktriangle}{d}_2 \ldots \overset{\blacktriangle}{d}_j \overset{\blacksquare}{e}_1 \ldots][\ldots \overset{\blacktriangle}{d}_{(j+1)} \ldots \overset{\blacktriangle}{d}_k \ldots] \ldots [\ldots \overset{\blacktriangle}{d}_s \ldots] \ldots .$$

The dottings of the distinct letters are split-shuffles of disjoint sets. We can expand a particular dotting and leave the others intact as dottings. We expand the d_i and the expression above is

$$= sign(T)\sum_{\sigma} sign(\sigma)[\dot{a}_1\dot{a}_2\dots d_{\sigma 1}\dots d_{\sigma j}\overset{\blacksquare}{e}_1\dots][\dots d_{\sigma(j+1)}\dots d_{\sigma k}\dots]\dots[\dots d_{\sigma s}\dots]\dots$$

where the sum is over the shuffles of the $(j, k-j, \dots, s-l)$ split of the d_j.

Writing the bracket sum as a sum of tableaux we have

$$= sign(T)\sum_{\sigma} sign(\sigma)sign(T_\sigma)\begin{pmatrix} a_1 & a_2 & \dots & d_{\sigma 1} & d_{\sigma 2} & \dots & d_{\sigma j} & e_1 & \dots \\ \cdot & \cdot & d_{\sigma(j+1)} & \cdot & \cdot & \cdot & d_{\sigma k} & \cdot & \cdot \\ & & & & \vdots & & & & \\ \cdot & \cdot & \cdot & \cdot & d_{\sigma s} & \cdot & \cdot & \cdot & \cdot \\ & & & & \vdots & & & & \end{pmatrix}.$$

where T_σ are the corresponding compact tableaux on letters $a, b, c, d_1, \dots, d_s, e, f, \dots$: i.e.

$$= sign(T)\sum_{\sigma} sign(\sigma)sign(T_\sigma)T_\sigma$$

$$= sign(T)\sum_{\sigma} sign(\sigma)sign(T_\sigma)\begin{pmatrix} \dots & X & \dots & d_{\sigma 1} & \dots & d_{\sigma j} & \dots & Y & \dots \\ \dots & Z & \dots & d_{\sigma(j+1)} & \dots & d_{\sigma k}\dots & W & \dots & \\ & & & & \vdots & & & & \\ \dots & U & \dots & d_{\sigma(l+1)} & \dots & d_{\sigma s} & \dots & V & \dots \\ & & & & \vdots & & & & \end{pmatrix}.$$

Now we show that for all σ the product $sign(T)sign(\sigma)sign(T_\sigma) = 1$. Suppose δ is the permutation that orders the vectors of the dotted bracket corresponding to T, so $sign(T) = sign(\delta)$. Let δ_σ be the permutations that order the vectors of the dotted brackets corresponding to the T_σ. Then clearly $\delta = \delta_\sigma\sigma$: thus $sign(T) = sign(\sigma)sign(T_\sigma)$ and $sign(T)sign(\sigma)sign(T_\sigma) = 1$ for all σ. So $T = \sum_\sigma T_\sigma$. ∎

We will now establish an identity in compact tableaux. First we define a split sum over a multiset. Let our split be a partition of a multiset whose elements are

from a linearly ordered set $(a, b, c, \dots)$. A shuffle of a particular split of the multiset is a permutation of the elements of the multiset such that each block of the split is ordered ($\leq$) by the linear order on the underlying set. If we pick certain entries of our tableau with repeated letters, the rows of the tableau effect a split. We will call the sum over all shuffles that net distinct summands the **multiset split-sum**. For example the tableau split-sum of the boldfaced letters of

$$\begin{pmatrix} a & a & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix}$$

is

$$\begin{pmatrix} a & a & c & d \\ b & b & c & d \end{pmatrix}+\begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix}+\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}+\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix}+\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

which we denote by $\sum_{\sigma} \sigma \begin{pmatrix} a & a & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix}$. Note that there is no consideration of the signature of the permutations, σ. There is some ambiguity over which shuffle caused certain terms to appear in the split sum. For example, the first term of the sum could be a consequence of the identity permutation on the original tableau or a consequence of the transposition of the c in the first row and the c in the second row. Because of a tableau signature convention we will not need to be concerned with which shuffle netted a particular term, only that every possible term appears in the sum. We can characterize the permutations of a multiset split-sum as the set of all shuffles that do not exchange copies of the same letter between rows, in the case where the split is between two rows.

Note also that if the multiset happens to have no repeated letters, we have a split of m distinct letter into k blocks of size i_k. Then the tableau split-sum will have $\binom{m}{i_1 \dots i_k}$ terms.

The identity we will prove will have the form of the van der Waerden syzygies on brackets. Split-sums with appropriate coefficients will equal zero. We will use boldface

to denote the set of letters to be shuffled in the identity. Note that, unlike dotted brackets, where dotting equates the expression with a bracket polynomial, boldfaced letters in a tableau have no intrinsic meaning. The boldfacing simply identifies letters to be shuffled.

THEOREM 3.4. *Suppose T is a rectangular compact tableau with n columns. Suppose further that in some two rows of T there are at least $n+1$ letters boldface and that if any letter of a row is boldface then all occurrences of this letter in the same row are boldface. Then we have the following identity:*

$$\sum_{\delta}(c_{\delta a}c_{\delta b}c_{\delta c}\cdots c_{\delta d})\ \delta\begin{pmatrix} & & & & \vdots & & & & \\ \dots & A & \dots & \mathbf{a} & \dots & \mathbf{b} & \dots \\ & & & & \vdots & & & & \\ \dots & \mathbf{c} & \dots & \mathbf{d} & \dots & E & \dots \\ & & & & \vdots & & & & \end{pmatrix}=0,$$

where the sum ranges over all multiset split shuffles of the boldfaced letters and the coefficients, $c_{\delta x}$, are determined relative to the effect of δ on the boldface letters as follows:

for the letter x, δ moves p x's into a row with j x's, $p \geq 0, j \geq 0$.

1) If the j x's are boldface, then $c_{\delta x} = 1$.

2) If the j x's are not boldface, then $c_{\delta x} = \binom{j+p}{j}$.

PROOF: For convenience we will prove the theorem for tableaux with two rows. Rows with no letters boldface are not affected by the syzygies. So in proving the theorem for two rows we will have proven it as stated for any number of rows.

We induct on m, the number of letters among those boldface that occur more than once in the tableau.

If $m = 0$, then all boldface letters appear once. The identity is just the usual van der Waerden syzygy [26] with the tableau signature convention absorbing the signs.

Suppose the theorem is true for $m = 1, 2, \ldots, r$. Suppose T is a rectangular tableau with two rows having certain letters in row 1 and certain letters in row 2 boldface such that if a particular letter is boldface, then every occurrence of that letter in the same row is boldface. Assume there are precisely $r+1$ letters among those boldface that occur more than once in the tableau and in particular there are $k > 1$ occurrences of the letter d, j in row 1 and $k-j$ in row 2, $k \geq j \geq 0$.

The tableau entries were defined to be from a multiset of a linearly ordered set, but we have not as yet used the order. By the signature convention we may permute entries within a row and the resulting tableau is equivalent to the same dotted bracket monomial. We use this fact to assume the entries of T are in a certain convenient order.

$$T = \begin{pmatrix} A & & & & X & d & d & \ldots & d \\ Y & d & d & \ldots & d & & & B & \end{pmatrix}.$$

Here the blocks A of row 1 and B of row 2 are the letters of T not boldfaced and blocks X of row 1 and Y of row 2 are the letters other than d which are boldface. For the sake of illustration we represented T with d's in both rows but it is not necessarily the case that both rows contain d's and if both do it is not necessary that the d's of both rows be boldface.

By Theorem 3.3,

$$(1) \qquad T = \sum_{\sigma} \begin{pmatrix} A & & X & d_{\sigma 1} & d_{\sigma 2} & \ldots & d_{\sigma j} \\ Y & & d_{\sigma(j+1)} & \ldots & d_{\sigma k} & & B \end{pmatrix} = \sum_{\sigma} T_{\sigma},$$

where the T_{σ} are tableaux on $\ldots, a, b, c, d_1, d_2, \ldots, d_k, e, \ldots$.

At this point we consider two cases separately:

1) both the j d_i's of row 1 and the $k-j$ d_i's of row 2 are boldface, $0 \leq j \leq k$.

2) the d_i's of only one of the rows are boldface.

In case 1 we peel the $\delta = I$ term of the summation off to obtain

$$(2) \qquad T = \begin{pmatrix} A & & & X & d_1 & d_2 & \ldots & d_j \\ Y & d_{j+1} & \ldots & d_k & & & B & \end{pmatrix} + \sum_{\sigma \neq I} T_{\sigma}.$$

The induction hypothesis applies to all the terms of the summation. We will use it on the first term and substitute ($\sum_\delta T_\delta = 0 \implies T_I = -\sum_{\delta \neq I} T_\delta$). The remaining terms of the summation in equation (2) will occur in the substituted summation with a negative sign and coefficient 1 (since the only letters that are shuffled are the d_i and they occur once each) and thus will be cancelled. We will be left with

$$T = -\sum_{\delta} c_\delta T_\delta, \tag{3}$$

where the sum is over certain shuffles, δ, of $a, b, c, d_1, d_2, \ldots, d_k, e, \ldots$. The identity shuffle, I, will not be among these. The coefficient $c_\delta = c_{\delta a} c_{\delta b} \cdots$ (note that $c_{\delta d_i} = 1$ since the letters d_i occur once). A typical summand looks like

$$c_\delta \begin{pmatrix} A & & X_\delta & & d_{\delta 1} & d_{\delta 2} & \cdots & d_{\delta j} & d_{\delta(j+1)} & \cdots & d_{\delta(j+p)} \\ & Y_\delta & & d_{\delta(j+p+1)} & \cdots & d_{\delta_k} & & & B & & \end{pmatrix}.$$

This term is obtained by the multiset shuffle, δ, on T_I. This shuffle fixes part of block X, say X'_δ, and fixes part of block Y, say Y''_δ, and shuffles X''_δ from X and Y'_δ from Y and perhaps some of the d_i, netting

$$c_\delta \begin{pmatrix} A & X'_\delta & & Y'_\delta & d_{\delta 1} & \cdots & d_{\delta(j+p)} \\ X''_\delta & Y''_\delta & d_{\delta(j+p+1)} & \cdots & d_{\delta k} & & B \end{pmatrix},$$

where $X'_\delta \bigcup Y'_\delta = X_\delta$ and $X''_\delta \bigcup Y''_\delta = Y_\delta$ (multiset unions).

Now if two terms of the sum agree in every entry that is not d_i, i.e. if $X_{\delta_1} = X_{\delta_2}$ and thus $Y_{\delta_1} = Y_{\delta_2}$, then $c_{\delta_1} = c_{\delta_2}$ since they shuffle the same set of letters that are not d_i, and hence $c_{\delta_1 a} = c_{\delta_2 a}, c_{\delta_1 b} = c_{\delta_2 b}$, etc. Suppose we have a particular term $c_\delta T_\delta$ as above. Consider the collection of all summands, tableaux $c_{\alpha_i} T_{\alpha_i}$, with $X_{\alpha_i} = X_\delta$. These have form

$$c_{\alpha_i} \begin{pmatrix} A & & X_\delta & d_{\alpha_i 1} & & d_{\alpha_i(j+p)} \\ Y_\delta & d_{\alpha_i(j+p+1)} & & d_{\alpha_i k} & B & \end{pmatrix}.$$

Clearly the collection of these terms consists exactly of all the $(j+p,\ k-(j+p))$-split shuffles of the d_i,

$$\left(\begin{array}{ccccccc} & A & & X_\delta & d_{\gamma 1} & & d_{\gamma(j+p)} \\ Y_\delta & & d_{\gamma(j+p+1)} & & d_{\gamma k} & B & \end{array}\right).$$

In case 1 we used the induction hypothesis on only one tableau of equation (2) so there were no duplications. By Theorem 3.1 this collection of summands is equivalent to

$$c_\delta \left(\begin{array}{cccccccc} & A & & X_\delta & d & d & & d \\ Y_\delta & & d & d & d & & B & \end{array}\right),$$

a tableau on letters $a, b, c, d, e, \ldots$ This tableau retains the coefficient, c_δ, from the induction hypothesis expansion, and $c_\delta = c_{\delta a} c_{\delta b} c_{\delta c} c_{\delta d_1} c_{\delta d_2} \cdots c_{\delta d_k} c_{\delta e} c_{\delta f} \ldots$, since as noted before the $c_{\delta_{d_i}} = 1$. So (3) then becomes $T = -\sum\limits_{\sigma} (c_{\delta a} c_{\delta b}\, c_{\delta a})\, T_\sigma$ and case 1 is proven.

In case 2 the d_i's of one row of T are not boldface. Without loss of generality we assume the d_i's of row 1 are not boldface. Recall $T = \sum_\sigma T_\sigma$. This time we will apply the induction hypothesis to each summand, obtaining

$T = \sum_\sigma (-\sum_\gamma c_\gamma T_\gamma) = -\sum_\delta c_\delta T_\delta$. If a particular term,

$$c_\delta T' = c_\delta \left(\begin{array}{ccccc} A & X_\delta & d_{\delta 1} & d_{\delta 2} & d_{\delta(j+p)} \\ Y_\delta & d_{\delta(j+p+1)} & & d_{\delta k} & B \end{array}\right),$$

appears then it is a consequence of an induction hypothesis split shuffle, γ, on one of the tableaux, T_σ, of equation (1),

$$c_\delta T' = c_\gamma \gamma \left(\begin{array}{ccccccc} A & & X & d_{\sigma 1} & d_{\sigma 2} & & d_{\sigma j} \\ Y & d_{\sigma(j+1)} & & d_{\sigma k} & & B & \end{array}\right), \text{ etc.}$$

Here γ shuffles some letters of X, say X''_δ, and some letters of Y, say Y'_δ, leaving X'_δ and Y''_δ of X and Y respectively fixed and moving p d_i's from row 2 to row 1. So

$$c_\delta T' = c_\delta \left(\begin{array}{cccccccc} A & X'_\delta & & Y'_\delta & d_{\delta 1} & d_{\delta 2} & & d_{\delta(j+p)} \\ X''_\delta & Y''_\delta & d_{\delta(j+p+1)} & & d_{\delta k} & & B & \end{array}\right).$$

Of course, γ shuffles the same number of letters from row 1 and row 2, in particular $|Y'_\delta| + p = |X''_\delta|$.

There is an induction hypothesis shuffle ρ on

$$T_I = \left(\begin{array}{cccccc} A & & X & d_1 & d_2 & d_j \\ Y & d_{j+1} & & & d_k & B \end{array}\right)$$

that exchanges X''_δ for Y'_δ and $d_{j+1}, \ldots, d_{j+p}$ i.e.

$$\begin{aligned} &\rho\left(\begin{array}{ccccccc} A & & X & d_1 & d_2 & & d_j \\ Y & d_{j+1} & & d_k & & B & \end{array}\right) \\ &= \left(\begin{array}{ccccccccc} A & & X' & & Y' & & d_1 & & d_{j+p} \\ X'' & Y'' & & d_{j+p+1} & & d_k & & B & \end{array}\right) \\ &= \left(\begin{array}{cccccc} A & X_\delta & & d_1 & \cdots & d_{j+p} \\ Y_\delta & d_{j+p+1} & \cdots & d_k & & B \end{array}\right). \end{aligned}$$

Now we show that we must have any tableau,

$$\left(\begin{array}{cccccc} A & X_\delta & & d_{\alpha 1} & \cdots & d_{\alpha(j+p)} \\ Y_\delta & d_{\alpha(j+p+1)} & \cdots & d_{\alpha k} & & B \end{array}\right)$$

occurring as $\gamma(T_\sigma)$ for some γ and σ, for any arbitrary $(j+p,\ k-(j+p))$-split shuffle. α, of the d_i with the letters other than d_i fixed. This tableau is a consequence of ρ acting on

$$T'' = \left(\begin{array}{cccc} A & X & d_{\alpha 1} & d_{\alpha j} \\ Y & d_{\alpha(j+1)} & d_{\alpha k} & B \end{array}\right).$$

The tableau T'' is necessarily among the T_σ of equation (1). So if we have one tableau.

$$c_\delta T_\delta = c_\delta \left(\begin{array}{ccccc} A & X_\delta & d_{\delta 1} & d_{\delta 2} & d_{\delta(j+p)} \\ Y_\delta & d_{\delta(j+p+1)} & & d_{\delta k} & B \end{array}\right),$$

then we must have all $\binom{k}{p}$ shuffles of the $(p, k-p)$-split of the d_i. It remains to show that we have the same number of occurrences of such terms so that we can gather these and apply Theorem 3.1 to realize them as a tableau in $\ldots, a, b, c, d, \epsilon, \ldots$.

For a particular tableau T'',

$$T'' = \begin{pmatrix} A & X_\delta & & d_{\delta 1} & d_{\delta 2} & \dots & d_{\delta(j+p)} \\ Y_\delta & d_{\delta(j+p+1)} & \dots & d_{\delta k} & & B & \end{pmatrix}$$

$$= \rho \begin{pmatrix} A & X & & d_{\sigma 1} & d_{\sigma 2} & \dots & d_{\sigma j} \\ Y & d_{\sigma(j+1)} & \dots & d_{\sigma k} & & B & \end{pmatrix}$$

where ρ is an induction hypothesis shuffle. Since the row 1 d_i are not boldface. ρ moves p d_i from row 2 to row 1. To obtain T'', j of the $j+p$ row 1 entries. $d_{\delta 1}, \dots, d_{\delta(j+p)}$, must be in row 1 prior to the shuffle. There are $\binom{p+j}{j}$ such tableaux in the equation (1) expansion and clearly for each of these there is a shuffle, ρ, that moves the remaining p d_i into row 1 as well as shuffling the appropriate X and Y letters. Therefore there will be $\binom{j+p}{j}$ occurrences of

$$c_\delta \begin{pmatrix} A & X_\delta & d_{\delta 1} & d_{\delta 2} & d_{\delta(j+p)} \\ Y_\delta & d_{\delta(j+p+1)} & & d_{\delta k} & B \end{pmatrix}$$

for each $(j+p,\ k-(j+p))$-split of the d_i, and fixed X_δ. So we gather these terms and use Theorem 3.3 to obtain $\binom{j+p}{j}$ occurrences of

$$c_\delta \begin{pmatrix} A & X_\delta & & d & d & & d \\ Y_\delta & d & d & & d & B & \end{pmatrix}.$$

Setting $c_{\delta d} = \binom{j+p}{j}$ we have proven case 2). ■

EXAMPLE 3.5. The identity on the boldface letters of

$$\begin{pmatrix} a & b & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix}$$

gives

$$\begin{matrix} a & b & c & d \\ b & b & c & d \end{matrix} + 1\cdot 2\cdot 1\cdot 1 \begin{matrix} a & b & b & d \\ b & c & c & d \end{matrix} + 1\cdot 2\cdot 1\cdot 2 \begin{matrix} a & b & b & c \\ b & c & d & d \end{matrix}$$

$$+1\cdot 3\cdot 1\cdot 2 \begin{matrix} a & b & b & b \\ c & c & d & d \end{matrix} + 1\cdot 1\cdot 1\cdot 2 \begin{matrix} a & b & c & c \\ b & b & d & d \end{matrix} = 0.$$

Equivalently,

$$\begin{matrix} a & b & c & d \\ b & b & c & d \end{matrix} = -2 \begin{matrix} a & b & b & d \\ b & c & c & d \end{matrix} - 4 \begin{matrix} a & b & b & c \\ b & c & d & d \end{matrix} - 6 \begin{matrix} a & b & b & b \\ c & c & d & d \end{matrix} - 2 \begin{matrix} a & b & c & c \\ b & b & d & d. \end{matrix}$$

This is the most common form of the identity in applications, where we substitute the sum on the right side of the equation for the tableau on the left side.

EXAMPLE 3.6.

$$\begin{pmatrix} a & a & \mathbf{e} & \mathbf{f} \\ b & b & d & e \\ \mathbf{c} & \mathbf{c} & \mathbf{d} & f \end{pmatrix} = - \begin{pmatrix} a & a & c & f \\ b & b & d & e \\ c & d & e & f \end{pmatrix} - \begin{pmatrix} a & a & d & f \\ b & b & d & e \\ c & c & e & f \end{pmatrix} - 2 \begin{pmatrix} a & a & c & e \\ b & b & d & e \\ c & c & e & f \end{pmatrix}$$

$$-2 \begin{pmatrix} a & a & d & e \\ b & b & d & e \\ c & c & f & f \end{pmatrix} - 2 \begin{pmatrix} a & a & c & c \\ b & b & d & e \\ d & e & f & f \end{pmatrix} - 2 \begin{pmatrix} a & a & c & d \\ b & b & d & e \\ c & e & f & f. \end{pmatrix}$$

The entries of our compact tableaux are from a linearly ordered set. We use the Roman alphabet. As we have noted the signature convention allows us to order the entries of the rows as we wish and so far we have been flexible. Now we adopt the convention of ordering the row entries in ascending order. We also order the rows lexicographically in ascending order, treating the rows as n-letter words. In the dotted bracket correspondence this is just a matter of commuting brackets in products along with the signature consideration. For commuting brackets over an even rank space, the signature convention absorbs sign changes but over an odd rank space commuting effects a sign change. With this convention we can now impose an order on the compact tableaux. If T_1 and T_2 are $n \times m$ compact tableaux on the same letters and w_1 and w_2 are words obtained by concatenating the rows of the respective T_i, row 1 followed by row 2, etc., then we say $T_1 \leq T_2$ if and only if $w_1 \leq w_2$ (the lexicographical order on the w_i). We can make this comparison only after adopting the convention of ordering the row entries and the rows of the tableau. From now on when we refer to tableaux we will assume that this convention is adopted unless explicitly stated otherwise.

We define a **standard tableau** to be a tableau whose row entries are ascending and whose column entries are strictly ascending.

EXAMPLE 3.7.

$\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}$ is a standard tableau.

$\begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix}$ is not a standard tableau.

LEMMA 3.8. *Suppose T is a nonstandard tableau with entry y in row i and entry x in the same column of row $j, j > i$ and with $x \leq y$. I.e.,*

$$T = \begin{pmatrix} & \vdots & \\ A & y & B \\ & \vdots & \\ C & x & D \\ & \vdots & \end{pmatrix}.$$

If we boldface y and all the entries of row i to the right of y together with all other occurrences of y in row i, and boldface x and all the entries of row j to the left of x together with all other occurrences of x in row j, and apply Theorem 3.4, then we realize T as a sum of tableaux $T = \sum_\delta \pm c_\delta T_\delta$ where $T_\delta < T$ for every δ.

PROOF:

$$T = \begin{pmatrix} & \vdots & \\ A & y & B \\ & \vdots & \\ C & x & D \\ & \vdots & \end{pmatrix}$$

By our row ordering convention if $b \in B$ and $c \in C$, then $c \leq x \leq y \leq b$. The shuffle. δ, exchanges certain boldface letters of T. Since the shuffles do not transpose the same letters between rows, the largest letter shuffled from row j is strictly less than the smallest letter shuffled from row i. The new row of T that replaces row i is smaller than row i of T. It is possible that the new rows obtained from rows i and j are smaller than rows that preceded rows i or j respectively, in T. The tableau row ordering convention may permute the new rows higher. In general, rows at level i or above are being replaced by smaller rows, so $T_\delta < T$. ∎

EXAMPLE 3.9.

$$\begin{matrix} a & a & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{matrix} = -\begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} - 2\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

The $\pm$ sign in the summation in Lemma 3.8 allows for signature variation due to row ordering permutations. This can be specified by the following term

$$-[\text{sign } (\rho\ )]^n,$$

where ρ is the permutation of the rows that orders the rows of the new tableau and n is the rank of the underlying vector space. The exponent accommodates the fact that row transpositions are n letter transpositions, so only in odd rank spaces do row transpositions effect sign changes. This refinement was not incorporated into Lemma 3.8 because the important aspect of the identity is that it decreases the terms of a tableau expression in the tableau order. We now use this to prove the following theorem.

THEOREM 3.10 (THE STRAIGHTENING ALGORITHM). *The standard compact tableaux form a vector space basis for the algebra of compact tableaux imposed by the dotted bracket correspondence, i.e. a basis for linear invariants of extensors.*

PROOF: If T is a nonstandard tableau, we apply Lemma 3.9, obtaining an equivalent sum of tableaux. We set aside those tableaux in the sum which are standard and apply the lemma again to those which are nonstandard. We continue this process iteratively. Since there are a finite number of tableaux on the letters of T having the same shape as T, and since the smallest tableau among these,

$$\begin{pmatrix} a & a & & \dots & & a & b & b & & \dots & b & c & \dots \\ c & \dots & c & d & d & & \dots & d & e & e & & \dots & \\ & & & & & & \vdots & & & & & & \end{pmatrix}$$

is standard, this process must end with T realized as a sum of standard compact tableaux.

It remains to show that the standard tableaux are independent. Suppose T_i are standard compact tableaux on letters $a, b, c, \ldots$ and $\sum \alpha_i T_i = 0$. The tableau-dotted bracket correspondence gives

$$0 = \sum_i \alpha_i T_i = \sum_i sign(T_i)\alpha_i[\dot{a}_1 \ldots][\dot{a}_j \ldots] \ldots .$$

Expanding the dottings,

$$= \sum_i sign(T_i)\alpha_i \sum_\sigma sign(\sigma)[a_{\sigma 1} \ldots][a_{\sigma j} \ldots] \ldots$$

which we can write as a sum of tableau on letters $a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}, \ldots$.

$$= \sum_i sign(T_i)\alpha_i \sum_\sigma sign(\sigma) sign(T_{\sigma_i}) T_{\sigma_i}.$$

The tableaux of this sum are standard. The independence of these tableaux in the standard basis theorem [15] for vector invariants implies that $\alpha_i = 0$, for all i. Therefore the standard tableau for extensor invariants are independent.

■

EXAMPLE 3.11. Straightening the previous example, we get:

$$\begin{matrix} a & a & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{matrix} =$$

$$- \begin{pmatrix} a & a & b & \mathbf{d} \\ \mathbf{b} & \mathbf{c} & \mathbf{c} & \mathbf{d} \end{pmatrix} - 2 \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$= 2 \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}$$

$$- 2 \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$= - \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix}$$

EXAMPLE 3.12: . The superbracket (Example 3.2) straightens as follows:

$$\begin{pmatrix} a & a & d & e \\ b & b & d & f \\ c & c & e & f \end{pmatrix} - \begin{pmatrix} a & b & d & d \\ a & c & e & e \\ b & c & f & f \end{pmatrix} = \left| \begin{pmatrix} a & a & b & b \\ c & c & d & e \\ d & e & f & f \end{pmatrix} \right.$$

$$- \begin{pmatrix} a & a & b & c \\ b & c & d & d \\ e & e & f & f \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & d & d & e \\ c & e & f & f \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & e \\ d & e & f & f \end{pmatrix}.$$

The equality of linear invariant functions can be well concealed by bracket or tableau syzygies. To determine whether two linear invariant functions of the same tensors are in fact equal we straighten the difference between the two functions.

EXAMPLE 3.13. Let

$$f(a_1a_2, b_1b_2, c_1c_2, d_1d_2) = [a_1a_2d_1d_2][b_1b_2c_1c_2] - [a_1a_2c_1c_2][b_1b_2d_1d_2]$$

and $g(a_1a_2, b_1b_2, c_1c_2, d_1d_2) = [a_1\overset{\bullet}{a_2}\overset{\blacktriangle}{b_1}\overset{\bullet}{d_1}][b_2c_1c_2\overset{\blacktriangle}{d_2}] - [a_1a_2b_1b_2][c_1c_2d_1d_2]$.

To test whether $f = g$, we straighten the equivalent tableau expression of $f - g$:

$$\begin{pmatrix} a & a & \mathbf{d} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & \mathbf{c} \end{pmatrix} - \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} + \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$= \left\{ - \begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} + \begin{pmatrix} a & a & c & d \\ b & b & c & d \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix} + \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} \right\}$$

$$- \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} + \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$= - \begin{pmatrix} a & a & c & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} + 2 \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} \right\} - \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2 \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} = 0.$$

In implementing the straightening algorithm we order the terms of the invariant to be straightened and do the iterative straightening on the largest compact tableau in the queue. This way we appreciate the cancellations that occur when we replace this term with a sum of smaller terms and we avoid repeating a straightening of a nonstandard tableau [24].

For a particular set of extensors in a particular rank space we can list the basis for the space of linear invariant functions. For example, in rank four the linear invariant functions of the two extensors $a_1a_2, b_1b_2, c_1c_2, d_1d_2$ are linear combinations of the functions

$$[a_1a_2b_1b_2][c_1c_2d_1d_2], [a_1a_2\overset{\bullet}{b}_1\overset{\blacktriangle}{c}_1][\overset{\bullet}{b}_2\overset{\blacktriangle}{c}_2d_1d_2], [a_1a_2c_1c_2][b_1b_2d_1d_2]$$

as it is easy to verify that the three tableaux,

$$\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}, \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}, \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix}$$

are the only 2×4 standard compact tableaux from the multiset $\{a, a, b, b, c, c, d, d\}$.

The celebrated hook length formula counts the number of standard tableaux of a particular shape with no repeated letters in the tableaux [1]. The number of standard tableaux of a given shape over a prescribed multiset is called a Kostka number. Discovering a formula for Kostka numbers is an open problem [20].

When all occurrences of a particular letter in a tableau polynomial are together in a row in each term, we say the tableau expression is **rectified** in this letter. Obviously all standard tableaux are rectified in the first letter of the linearly ordered tableau entries (typically the letter a). We can extend this definition of rectified to bracket polynomials saying the polynomial is rectified in a particular extensor. When a function that is an invariant function of extensors has a realization (either tableau or bracket polynomial) that is rectified in all of its arguments (letters or extensors) we say the realization is rectified.

COROLLARY 3.14. *A multilinear invariant function of extensors can be rectified in any of its tensor arguments.*

PROOF: Suppose we are given a bracket polynomial that is multilinear invariant function of extensors. By Theorem 2.5 we can construct a dotted bracket expression for this invariant. From this we can realize our invariant as a tableau expression. It is arbitrary which letter of the tableaux we associate with extensor arguments of the function. If we use the first letters of the Roman alphabet in the tableaux we can associate the a's with any extensor we wish. We associate it with the extensor we wish to rectify, and then straighten the tableau. The tableaux are rectified in the a's and the corresponding bracket polynomial is rectified in the extensor associated with the a. ■

COROLLARY 3.15. *Suppose that f is a multilinear function of vectors $a_1, \ldots, a_{k_1}, b_1, \ldots, b_{k_2}, \ldots, z_1, \ldots, z_{k_\ell}$ that can be realized as a dotted bracket expression where the distinct split sums of each expansion are dottings of $a_1, \ldots . a_{k_1}$,; $b_1, \ldots, b_{k_2}$,; … and $z_1, \ldots, z_{k_\ell}$ respectively, then f is a multilinear function of the tensors $a_1 \ldots a_{k_1}$, $b_1 \ldots b_{k_2}, \ldots, z_1 \ldots z_{k_\ell}$.*

PROOF: We can rectify f in $a_1, \ldots, a_{k_1}$. That is, we can realize f as

$$\sum [a_1 \ldots a_{k_1} \ldots] \ldots [\qquad\qquad].$$

Then clearly f is a multilinear function of $a_1 a_2 \ldots a_{k_1}$. Similarly f is multilinear in the extensor obtained by joining any of the vectors in its dotting splits.

■

If we have a tableaux corresponding to a multilinear invariant of tensors we can pick a particular letter and a particular row of the tableau and use the tableau

identities to move this letter into the row. The syzygy that boldfaces the letter and the all letters of the desired row effects this. For example

$$\begin{array}{cccc} a & a & c & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & \mathbf{e} \\ d & e & f & f \end{array} = -\left(\begin{array}{cccc} a & a & b & c \\ b & c & d & e \\ d & e & f & f \end{array} + 2 \begin{array}{cccc} a & a & c & c \\ b & b & d & e \\ d & e & f & f \end{array} + \begin{array}{cccc} a & a & c & e \\ b & b & c & d \\ d & e & f & f \end{array} \right).$$

moves a d into the second row of each tableau on the right side of the equation. Similarly a second copy of this letter can be moved into the same row and the first copy will remain in the row in each term of the substitution. So we can iteratively gather all copies of a particular letter in a particular row. Extending the example above by boldfacing the d in the third row and all letters of the second row we get

$$= \begin{array}{cccc} a & a & b & c \\ c & d & d & e \\ b & c & f & f \end{array} + \begin{array}{cccc} a & a & b & c \\ b & d & d & e \\ c & e & f & f \end{array} + 2 \begin{array}{cccc} a & a & b & c \\ b & c & d & d \\ e & e & f & f \end{array} + \begin{array}{cccc} a & a & c & c \\ b & d & d & e \\ b & e & f & f \end{array}$$

$$+2 \begin{array}{cccc} a & a & c & c \\ b & b & d & d \\ e & e & f & f \end{array} + \begin{array}{cccc} a & a & c & e \\ b & c & d & d \\ b & e & f & f \end{array} + \begin{array}{cccc} a & a & c & e \\ b & b & d & d \\ c & e & f & f, \end{array}$$

which is rectified in the d's in the second row.

THEOREM 3.16. *A multilinear invariant function of tensors whose bracket realization is of bracket degree n, can be rectified simultaneously in n of its tensor arguments.*

PROOF: We can continue the process outlined above rectifying a different letter in a different row. Since the syzygy we are using moves only the particular letter of interest and letters of the row of interest, it will not disturb letters previously rectified in other rows. This algorithm will rectify one letter (extensor) in each row of the tableau.

■

EXAMPLE 3.17. The superbracket rectified in letters a, b, and c in rows 1, 2, and 3 respectively is

$$\begin{array}{cccc} a & a & d & e \\ b & b & d & f \\ c & c & e & f \end{array} + \begin{array}{cccc} a & a & d & d \\ b & b & e & f \\ c & c & e & f \end{array} + 2 \begin{array}{cccc} a & a & d & d \\ b & b & f & f \\ c & c & e & e \end{array} + \begin{array}{cccc} a & a & d & e \\ b & b & f & f \\ c & c & d & e \end{array}$$

$$+ \begin{array}{cccc} a & a & d & f \\ b & b & d & f \\ c & c & e & e \end{array} + \begin{array}{cccc} a & a & d & f \\ b & b & e & f \\ c & c & d & e. \end{array}$$

EXAMPLE 3.18. There are three standard 2×4 tableaux containing four pairs of letters;

$$\begin{array}{cccc} a & a & b & b \\ c & c & d & d, \end{array} \quad \begin{array}{cccc} a & a & b & c \\ b & c & d & d, \end{array} \quad \text{and} \quad \begin{array}{cccc} a & a & c & c \\ b & b & d & d. \end{array}$$

$$\begin{array}{cccc} a & a & b & c \\ b & c & d & d \end{array} = \begin{array}{cccc} a & a & d & d \\ b & b & c & c \end{array} - \begin{array}{cccc} a & a & b & b \\ c & c & d & d \end{array} - \begin{array}{cccc} a & a & c & c \\ b & b & d & d. \end{array}$$

So the tableaux

$$\begin{array}{cccc} a & a & b & b \\ c & c & d & d, \end{array} \quad \begin{array}{cccc} a & a & c & c \\ b & b & d & d, \end{array} \quad \text{and} \quad \begin{array}{cccc} a & a & d & d \\ b & b & c & c \end{array}$$

form a vector space basis of all linear invariants of 4 step–two tensors in rank–four, with a bracket realization of degree 2. Therefore any linear invariant function of four extensors of step–two in projective rank–four space equals a completely rectified expression. It is not always possible to completely rectify an invariant. The superbracket cannot be completely rectified.

There is another interesting observation made evident by the straightening algorithm and the fact that it is arbitrary how tableau letters are associated with extensor arguments of linear invariant functions. The number of tableaux of a particular shape with entries from a multiset with k_1 occurrences of one letter, k_2 occurrences of a second letter, k_3 occurrences of a third letter, etc., is independent of which letter occurs k_1 times and which occurs k_2 times, etc. For instance, the number of 3×4 standard tableaux on letters $\{a, a, a, b, b, c, c, d, e, e, e, f\}$ e.g.,

$$\begin{pmatrix} a & a & a & b \\ b & c & c & d \\ e & e & e & f, \end{pmatrix} \quad \begin{pmatrix} a & a & a & b \\ b & c & d & e \\ c & e & e & f, \end{pmatrix} \quad \begin{pmatrix} a & a & a & c \\ b & b & c & e \\ d & e & e & f, \end{pmatrix} \ldots$$

is the same as the number of standard tableaux on letters $\{a, a, b, b, b, c, d, d, d, e, f, f\}$,

e.g.,

$$\begin{pmatrix} a & a & b & b \\ b & c & d & d \\ d & e & f & f, \end{pmatrix} \quad \begin{pmatrix} a & a & b & c \\ b & b & d & e \\ c & d & f & f, \end{pmatrix} \quad \begin{pmatrix} a & a & b & d \\ b & b & c & e \\ d & d & f & f, \end{pmatrix} \cdots$$

In general the count depends on the shape and the partition, $\{k_1, k_2, k_3, \ldots\}$ of nm, the number of letter-places in the tableaux.

CHAPTER 4

SIGN PATTERNS OF LINE ARRANGEMENTS IN PROJECTIVE RANK-FOUR

Plucker coordinates give a homogeneous coordinatization of lines. Given the projective coordinates of two points in 3-space (projective rank-4) the Plucker coordinates of the line determined by the points are the six 2×2 minors of the point coordinates. The minors can be presented in various orders. We adopt the following familiar convention:

$$(m_{1,4}, m_{2,4}, m_{3,4}, m_{2,3}, m_{3,1}, m_{1,2}).$$

For example, the points $(1,0,1,1)$ and $(2,3,-1,1)$ determine a line with Plucker coordinates:

$$\left(\begin{vmatrix}1 & 2\\ 1 & 1\end{vmatrix}, \begin{vmatrix}0 & 3\\ 1 & 1\end{vmatrix}, \begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}, \begin{vmatrix}0 & 3\\ 1 & -1\end{vmatrix}, \begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}, \begin{vmatrix}1 & 2\\ 0 & 3\end{vmatrix}\right) = (-1,-3,2,-3,3,3).$$

The Plucker coordinate vectors are elements of $\Lambda^2(R^4)$ and we refer to them as decomposable two-vectors. They can be interpreted as directed lines. The vectors $(v_1, v_2, v_3, v_4, v_5, v_6)$ of rank-6 projective space which are Plucker coordinates of lines are exactly those which satisfy the following quadratic condition

$$v_1v_4 + v_2v_5 + v_3v_6 = 0.$$

The vectors of $\Lambda^2(R^4)$ which do not satisfy the quadratic condition also have a physical interpretation, screw motions, i.e. rotations about a fixed axis together with a translation along the axis [3].

For $l_1 = (x_1, x_2, x_3, x_4, x_5, x_6)$ and $l_2 = (y_1, y_2, y_3, y_4, y_5, y_6)$ we define the following bilinear form on the space of lines and screws:

$$l_1 \cdot l_2 = x_1y_4 + x_2y_5 + x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3.$$

This is called the reciprocal product. If l_1 and l_2 are lines, then given two points X_1 and X_2 of l_1 which determine the Plucker coordinates and similarly two points of l_2, Y_1 and Y_2, the bilinear form is the Laplace expansion by the 2×2 minors of the 4×4 matrix whose column vectors are the coordinates of the four points. We use the notation $[l_1, l_2]$ for this form and refer to it as the bracket of l_1 and l_2.

Lines are precisely the two–vectors l with $[l, l] = 0$. Given two lines l_1 and l_2, $[l_1, l_2] = 0$ implies that the lines intersect.

Armed with this notation we examine some nonlinear invariants of lines in rank–4 projective space (invariants of two–extensors). These are **sign invariants**, functions whose signature is invariant under transformations on the underlying vector space.

THEOREM 4.1. *For l_1, l_2, and l_3 arbitrary lines of a regulus, sign $([l_1, l_2][l_1, l_3][l_2, l_3])$ is independent of the choice of three regulus lines.*

PROOF: We must show that if l_4, l_5, and l_6 are three arbitrary lines of the regulus generated by l_1, l_2, and l_3, then sign $([l_4, l_5][l_4, l_6][l_5, l_6]) =$ sign $([l_1, l_2], [l_1, l_3][l_2, l_3])$. We show that if we replace any one of l_1, l_2, or l_3 with l_4, a line of $\langle l_1, l_2, l_3 \rangle$, the signature is preserved. This implies that the signature is preserved under iterative replacement and hence implies the theorem.

First we note that we can obtain generators l_1, l_2, and l_3 such that $[l_1, l_2], [l_1, l_3]$, and $[l_2, l_3]$ agree in sign. If they do not initially agree, then two of the three brackets must have the same sign and these two have a line, l_i, in common. Replacing l_i with $-l_i$ in the generators switches the sign of the two commonly signed brackets to agree with the third bracket. E.g. $[l_1, l_2] > 0$, $[l_1, l_3] < 0$, and $[l_2, l_3] < 0 \Longrightarrow [l_1, l_2] > 0$, $[l_1, -l_3] > 0$, and $[l_2, -l_3] > 0$. (Note that sign $([l_1, l_2][l_1, l_3][l_2, l_3]) =$ sign $([l_1, l_2][l_1, -l_3][l_2, -l_3])$ so the signature of the regulus is preserved with this

switch.) We prove the theorem for the case $[l_i, l_j] > 0$, $i, j \in \{1,2,3\}$. An adjustment of this proof establishes the case $[l_i, l_j] < 0$, $i, j \in \{1,2,3\}$.

For convenience we scale the homogeneous coordinates of l_1, l_2, and l_3

$$l_1' = \sqrt{\frac{[l_2,l_3]}{[l_1,l_2][l_1,l_3]}}\, l_1,\ l_2' = \sqrt{\frac{[l_1,l_3]}{[l_1,l_2][l_2,l_3]}}\, l_2,$$

$$\text{and } l_3' = \sqrt{\frac{[l_1,l_2]}{[l_1,l_3][l_2,l_3]}}\, l_3.$$

Now we have generators l_1', l_2', and l_3' with $[l_1', l_2'] = [l_1', l_3'] = [l_2', l_3'] = 1$.

If l_4 is a line of the regulus of l_1', l_2', and l_3' then up to scaling $l_4 = l_1' + \alpha l_2' + \beta l_3'$, $\alpha, \beta \neq 0$.

$$[l_4, l_4] = 0 = 2\alpha + 2\beta + 2\alpha\beta \Longrightarrow \alpha + \beta + \alpha\beta = 0.$$

$$[l_4, l_1'] = \alpha + \beta.$$

$$[l_4, l_2'] = 1 + \beta.$$

$$[l_4, l_3'] = 1 + \alpha.$$

$$[l_4, l_1'][l_4, l_2'] = \alpha + \beta + \alpha\beta + \beta^2 = \beta^2 > 0.$$

$$[l_4, l_1'][l_4, l_3'] = \alpha + \beta + \alpha\beta + \alpha^2 = \alpha^2 > 0.$$

$$[l_4, l_2'][l_4, l_3'] = \alpha + \beta + \alpha\beta + 1 = 1 > 0.$$

Therefore $[l_4, l_i'][l_4, l_j'][l_i', l_j'] > 0$, $i, j \in \{1,2,3\}$, $i \neq j$.

The adjustment for the case $[l_i, l_j] < 0$ is to insert a $-$ sign under the radicals in the definitions of the l_i' and account for the signature changes effected by $[l_i', l_j'] = -1$. That $[l_4, l_i'][l_4, l_j'][l_i', l_j'] < 0$, $i, j \in \{1,2,3\}$ follows. ■

We call $\text{sign}([l_1, l_2][l_1, l_3][l_2, l_3])$ the **orientation** of the regulus spanned by l_1, l_2 and l_3.

The following theorem defines a sign invariant. A less specific version of this invariant appears in Forder [12].

THEOREM 4.2. *Let l_1, l_2, l_3, and l_4 be four mutually skew lines. The determinant of the 4×4 matrix $([l_i, l_j])$ is a sign invariant with the following interpretation:*

i) *If $\det([l_i, l_j]) = 0$, then l_4 is tangent to the regulus generated by l_1, l_2, and l_3.*

ii) *If $\det([l_i, l_j]) > 0$, then l_4 meets the regulus of l_1, l_2, and l_3 in two points.*

iii) *If $\det([l_i, l_j]) < 0$, then l_4 does not intersect the regulus of l_1, l_2, and l_3.*

(Note these conditions are symmetric in the four lines, l_1, l_2, l_3, and l_4.)

PROOF: The lines of the regulus of l_1, l_2, and l_3 have the following parameterization

$$l_\alpha = [l_2, l_3]\, l_1 + \alpha[l_1, l_3]\, l_2 - \frac{\alpha}{1+\alpha}\, [l_1, l_2]\, l_3.$$

This parameterization is derived from the quadratic condition on an arbitrary linear combination of l_1, l_2, l_3, $l = \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3$. The scaling $l = [l_2, l_3] l_1 + \alpha[l_1, l_3] l_2 + \beta[l_1, l_2] l_3$ nets $[l, l] = 0 \iff \beta = -\dfrac{\alpha}{1+\alpha}$.

Line l_4 meets the regulus means there is an α such that $[l_4, l_\alpha] = 0$.

$[l_4, l_\alpha] = [l_2, l_3][l_1, l_4] + \alpha[l_1, l_3][l_2, l_4] + \dfrac{\alpha}{1+\alpha}\, [l_1, l_2][l_4, l_3]$.

$[l_4, l_\alpha] = 0$ implies

$$\frac{1}{l+\alpha}([l_1, l_3][l_2, l_4]\alpha^2 + ([l_2, l_3][l_1, l_4] + [l_1, l_3][l_2, l_4] - [l_1, l_2][l_3, l_4])\alpha + [l_2, l_3][l_1, l_4]) = 0.$$

The expression in parentheses above is a quadratic in α, whose discriminant is exactly $\det([l_i, l_j])$ and the interpretations follow. E.g. if the determinant (discriminant) is positive, then there are two real α such that $[l_4, l_\alpha] = 0$ and so l_4 meets the regulus of l_1, l_2, and l_3 in two points. ■

Signature considerations are an integral part of subjects such as oriented matroids. Realizability problems can often be posed in terms of sequences of signatures. Eli Goodman suggested the following problem that lent itself to some of the methods developed here.

Given a set of directed lines $l_1, l_2, \ldots, l_k$ in rank-four projective space, a directed line l, skew to the k lines, generates a sequence, $\{\text{sign}([l, l_1]), \text{sign}([l, l_2]), \ldots, \text{sign}([l, l_k])\}$. Under what conditions can we find 2^k lines that generate all 2^k possible sign sequences?

Suppose we have some conditions on a set of lines which are independent of orientation; e.g. four skew lines or three coplanar lines. Suppose we are given a fixed sign sequence $\{s_1, s_2, \ldots, s_k\}$. If we can establish that for an arbitrary set of k lines $\{l_1, l_2, \ldots, l_k\}$ subject to our conditions, there is always a line l such that $\{\text{sign}([l, l_1]), \ldots, \text{sign}([l, l_k])\} = \{s_1, s_2, \ldots, s_k\}$, then, in fact, for an arbitrary sequence of k lines subject to our conditions there is such a line l for each of the 2^k sign sequences. For example, if for an arbitrary three skew lines, l_1, l_2, l_3 we can always find a line l with $[l, l_i] > 0$, then for three skew lines we can find a line for each of the 2^3 sign sequences. To obtain the sequence $\{-, +, -\}$, the line l' with $[l', -l_1] > 0$, $[l', l_2] > 0$, and $[l', -l_3] > 0$ gives $\{\text{sign}([l', l_i]) : i = 1, 2, \text{ or } 3\} = \{-, +, -\}$.

It is evident that a dependency among k lines will effect a dependency in the sign sequence that may prevent certain sequences, so we confine our attention to independent sets of k lines.

Lemma 4.3. *Given three skew lines, l_1, l_2, and l_3, there exists l such that $[l, l_i] > 0$, $i = 1, 2$, or 3.*

Proof: An arbitary line, l', skew to l_1, l_2, and l_3, can be oriented so that $[l', l_i] > 0$ for at least two of the three lines. Assume $[l', l_1] > 0$ and $[l', l_2] > 0$. If $[l', l_3] > 0$, then we are done. If not, let l'' be a transversal of l', l_1, and l_2 that does not intersect l_3. Such a line is possible so long as l' is not in the regulus of l_1, l_2, and l_3.

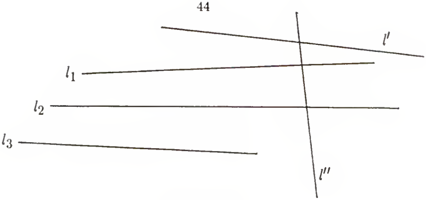

Figure 4.1

Any combination $l' + \alpha l''$ of the intersecting lines l' and l'' satisfies the quadratic condition and thus is a line.

Now orient l'' so that $[l'', l_3] > 0$. Let $l_0 = l' + \alpha l''$, with α scaled so that $[l_0, l_3] = [l', l_3] + \alpha[l'', l_3] > 0$. Then $[l_0, l_i] > 0$ for $i = 1, 2$, or 3. ∎

LEMMA 4.4. *Given four skew lines, l_1, l_2, l_3, and l_4, there exists a line, l, with $[l, l_i] > 0$, $i = 1, 2, 3$, or 4.*

PROOF: Lemma 4.3 guarantees a line, $l_0 = l' + \alpha l''$, with $[l_0, l_i] > 0$, $i = 1, 2$, or 3 (l' and l'' as above). We need a transversal of l_0, l_1, l_2, and l_3, that does not intersect l_4. For $i = 1, 2, 3, 0$,

$$\det([l_i, l_j]) = \begin{vmatrix} 0 & [l_1, l_2] & [l_1, l_3] & [l_1, l'] \\ [l_2, l_1] & 0 & [l_2, l_3] & [l_2, l'] \\ [l_3, l_1] & [l_3, l_2] & 0 & [l_3, l'] + \alpha[l_3, l''] \\ [l', l_1] & [l', l_2] & [l', l_3] + \alpha[l'', l_3] & 0 \end{vmatrix}.$$

So $\lim_{\alpha \to \infty} \det([l_i, l_j]) = +\infty$. So we can scale α so that for $l_0 = l' + \alpha l''$, $[l_0, l_k] > 0$, $k = 1, 2$, or 3, and $\det([l_i, l_j]) > 0$, for $i, j \in \{0, 1, 2, 3\}$. Then by Theorem 4.2, l_0 meets the regulus of lines l_1, l_2, and l_3 (in two points). Meeting the regulus means there is a line l_t of the conjugate regulus that is a transversal of l_0, l_1, l_2, and l_3. Moreover with the freedom to scale α we may assume l_t does not meet l_4.

Let l_0 be a line with $[l_0, l_i] > 0$, $i = 1, 2, 3$, (Lemma 4.3). Let l_t be a transversal of l_0, l_1, l_2, and l_3 that does not intersect l_4, as established above.

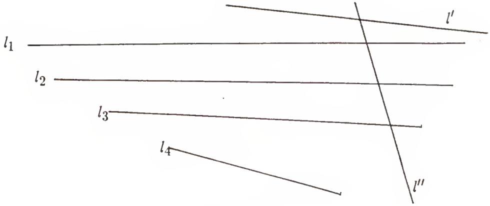

Figure 4.2

We may orient l_t so that $[l_t, l_4] > 0$. Then $l = l_0 + \alpha l_t$ with α appropriately scaled gives a line l with $[l, l_i] > 0$ for $i = 1, 2, 3$, or 4. ■

We have had occasion to examine families of lines that were the span of certain fixed sets of lines. The regulus is an example and there are other families generated by three lines (all lines through a given point for another example). The span of five independent lines is called a line complex [6]. A singular line complex is a special instance. A singular line complex is a line together with all lines that intersect this line. We can make an observation about five lines of a singular complex in the context we are considering.

LEMMA 4.5. *Given five lines, $l_1, l_2, \ldots, l_5$, of a singular line complex, there is a line, l, such that $[l, l_i] > 0$, $i = 1, 2, 3, 4$, or 5.*

PROOF: The span of the five lines being a singular complex means that there is a line, l', in the span that intersects all lines of the complex. In our notation, $\langle l_1, l_2, l_3, l_4, l_5 \rangle^{\perp} = \langle l' \rangle$, with $l' \in \langle l_1, l_2, l_3, l_4, l_5 \rangle$ and $[l', l'] = 0$.

We can always find a two-vector x, such that $[x, l_i] > 0$ or $i = 1, 2, 3, 4$, or 5, since

$$\begin{bmatrix} [l_1, x] \\ \vdots \\ [l_5, x] \end{bmatrix} = \begin{bmatrix} + \\ \vdots \\ + \end{bmatrix}$$

means

$$\begin{bmatrix} l_{11} & l_{12} & \dots & l_{16} \\ \vdots & & & \\ l_{51} & l_{52} & \dots & l_{56} \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} + \\ + \\ + \\ + \\ + \end{bmatrix},$$

where $x = (x_1, x_2, \dots, x_6)$. This is an overdetermined linear system. We cannot be certain yet that the solution satisfies the quadratic condition, i.e. that it is a line.

We assume x is a screw (if not we are done) such that $[x, l_i] > 0$, $i = 1, 2, 3, 4$ or 5. We may further assume that $[x, l'] \neq 0$; if this is not the case we can pick y such that $[y, l'] \neq 0$, and then scale x so that $[\alpha x + y, l_i] > 0$, $i = 1, 2, 3, 4$, or 5 and $[\alpha x + y, l'] \neq 0$.

Now we scale l' so that $2[x, l'] = -[x, x]$. Then $l = x + l'$ is a line since $[l, l] = [x, x] + 2[x, l'] + [l', l']$. Moreover $[l, l_i] = [x, l_i] > 0$ for $i = 1, 2, 3, 4$, and 5. ∎

EXAMPLE 4.6: For the following five independent lines, $l_1, \dots, l_5$, there does not exist a line l such that $[l, l_i] > 0$, $i = 1, 2, 3, 4, 5$

$$l_1 = (-1, -2, 0, -4, 2, 0)$$

$$l_2 = (4, 6, 8, 9, -6, 0)$$

$$l_3 = (-1, -1, 0, -1, 1, 0)$$

$$l_4 = (1, 1, 3, 2, 1, -1)$$

$$l_5 = (1, 2, -5, 1, 2, 1).$$

To establish this we get the two-vector x, such that $[x, l_i] = 0$, $i = 1, 2, 3, 4$, and 5 (i.e. $\langle l_1, \dots, l_5 \rangle^{\perp} = \langle x \rangle$) and we get the five two-vectors a_i such that

$$[a_i, l_j] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$x = (0, 1, 11, 0, 1, 3)$$

$$a_1 = (-2, .4265, -2.1912, -3, 4.5735, .2206)$$

$$a_2 = (-1, -.2941, -1.2353, -2, 2.7059, .1176)$$

$$a_3 = (-2, -4.118, -3.1294, -6, 6.5882, .2647)$$

$$a_4 = (0, .1912, -.3971, 0, .1912, .0735)$$

$$a_5 = (0, .1471, .1176, 0, .1471, .0588)$$

Now $[a_i, a_j] > 0$ for all i and j, and $[x, x] > 0$. The two-vectors $a_1, \ldots, a_5, x$ are a basis of the line-screw space $\Lambda^2(R^4)$. An arbitrary two-vector $l = \sum_{i=1}^{5} \alpha_i a_i + \alpha_6 x$.

For our five lines, $l_1, \ldots, l_5$, $[l, l_i] > 0$ implies that $\alpha_i > 0$, $i = 1, 2, 3, 4, 5$. But then $[l, l] = \sum_{i,j \in \{1,2,\ldots,5\}} \alpha_i \alpha_j [a_i a_j] + \alpha_6^2 [x, x] > 0$ so l is not a line. There is no line, l, such that $[l, l_i] > 0$ for $i = 1, 2, 3, 4$ and 5.

An interesting and useful perspective of the line-screw space is as a quadratic space. For vectors v_1 and v_2, $[v_1, v_2] = v_1 q v_2^t$, where q is the bilinear form with matrix

$$\begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}$$

over the standard basis.

Lines (i.e. vectors satisfying the quadratic condition) are the isotropic vectors of this form. For a given basis $b_1, b_2, b_3, b_4, b_5, b_6$, vector $v = \sum_{i=1}^{6} \alpha_i b_i$, and $[v, v] = \sum_{i,j \in \{1,2,\ldots,6\}} \alpha_i \alpha_j [b_i b_j]$ which equals the sum of the entries of the 6×6 matrix. $([\alpha_i b_i, \alpha_j b_j])$, the matrix of the form q in the basis $\alpha_1 b_1, \alpha_2 b_2, \ldots, \alpha_6 b_6$. In the example the matrix of q over $a_1, \ldots, a_5, x$,

$$\begin{bmatrix} [a_1, a_1] & [a_1, a_2] & \ldots & [a_1, a_5] & 0 \\ \vdots & & & & \vdots \\ [a_5, a_1] & [a_5, a_2] & \ldots & [a_5, a_5] & 0 \\ 0 & & \ldots & 0 & [x, x] \end{bmatrix}$$

has all nonnegative entries. For $l' = \alpha_1 a_1 + \ldots + \alpha_5 a_5 + \alpha_6 x$, $[l', l_i] = \alpha_i$. Then $[l', l_i] > 0$ implies that $\alpha_i > 0$ for $i = 1, 2, 3, 4, 5$. But then the matrix of q over $\alpha_1 a_1, \ldots, \alpha_5 a_5, \alpha_6 x$ has all positive entries and thus $[l', l'] \neq 0$. The following observations make this situation easier to detect.

Lemma 4.7. *Let lines $l_1, \ldots, l_5$ be such that $\langle l_1, \ldots, l_5 \rangle$ is a nonsingular line complex, and $\langle l_1, \ldots, l_5 \rangle^{\perp} = \langle v \rangle$, and five vectors $a_1, \ldots, a_5$ such that: $[a_i, l_j] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. Let L be the matrix of q over basis $l_1, \ldots, l_5, v$. Let A be the matrix of q over basis $a_1, \ldots, a_5, \frac{v}{[v,v]}$. Then $L^{-1} = A$.*

Proof:

$$L = \begin{bmatrix} l_{11} & l_{12} & \ldots & l_{16} \\ \vdots & & & \vdots \\ l_{51} & l_{52} & \ldots & l_{56} \\ v_1 & v_2 & \ldots & v_6 \end{bmatrix} \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & \ldots & l_{16} \\ \vdots & & & \vdots \\ l_{51} & l_{52} & \ldots & l_{56} \\ v_1 & v_2 & \ldots & v_6 \end{bmatrix}^t$$

$$= \begin{bmatrix} & [l_i, l_j] & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & [v, v] \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{16} \\ \vdots & & & \vdots \\ a_{51} & a_{52} & \ldots & a_{56} \\ v_1 & v_2 & \ldots & v_6 \end{bmatrix} \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{16} \\ \vdots & & & \vdots \\ a_{51} & a_{52} & \ldots & a_{56} \\ v_1 & v_2 & \ldots & v_6 \end{bmatrix}^t$$

$$= \begin{bmatrix} & [a_i, a_j] & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1/[v, v] \end{bmatrix}.$$

The defining condition on the a_i gives:

$$\begin{bmatrix} l_{11} & \ldots & l_{16} \\ \vdots & & \vdots \\ l_{51} & \ldots & l_{56} \\ v_1 & \ldots & v_6 \end{bmatrix} \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{16} \\ \vdots & & \vdots \\ a_{51} & \ldots & a_{56} \\ v_1 & \ldots & v_6 \end{bmatrix}^t = I_6.$$

Transposing we get

$$\begin{bmatrix} a_{11} & \dots & a_{16} \\ \vdots & & \vdots \\ a_{51} & \dots & a_{56} \\ v_1 & \dots & v_6 \end{bmatrix} \begin{bmatrix} 0 & I_3 \\ & \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} l_{11} & \dots & l_{16} \\ \vdots & & \vdots \\ l_{51} & \dots & l_{56} \\ v_1 & \dots & v_6 \end{bmatrix}^t = I_6.$$

So

$$\begin{bmatrix} 0 & I_3 \\ & \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} l_{11} & \dots & l_{16} \\ \vdots & & \vdots \\ l_{51} & \dots & l_{56} \\ v_1 & \dots & v_6 \end{bmatrix}^t = \begin{bmatrix} a_{11} & \dots & a_{16} \\ \vdots & & \vdots \\ a_{51} & \dots & a_{56} \\ v_1 & \dots & v_6 \end{bmatrix}^{-1}.$$

Thus $LA =$

$$\begin{bmatrix} l_{11} & \dots & l_{16} \\ \vdots & & \vdots \\ l_{51} & \dots & l_{56} \\ v_1 & \dots & v_6 \end{bmatrix} \begin{bmatrix} 0 & I_3 \\ & \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} l_{11} & \dots & l_{16} \\ \vdots & & \vdots \\ l_{51} & \dots & l_{56} \\ v_1 & \dots & v_6 \end{bmatrix}^t \times$$

$$\begin{bmatrix} a_{11} & \dots & a_{16} \\ \vdots & & \vdots \\ a_{51} & \dots & a_{56} \\ \frac{v_1}{[v,v]} & \dots & \frac{v_6}{[v,v]} \end{bmatrix} \begin{bmatrix} 0 & I_3 \\ & \\ I_3 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{16} \\ \vdots & & \vdots \\ a_{51} & \dots & a_{56} \\ \frac{v_1}{[v,v]} & \dots & \frac{v_6}{[v,v]} \end{bmatrix}^t = I_6.$$

■

For L and A above

$$LA = \begin{bmatrix} & [l_i, l_j] & & 0 \\ & & & \vdots \\ & & & 0 \\ & & & \\ 0 & \dots & 0 & [v,v] \end{bmatrix} \begin{bmatrix} & [a_i, a_j] & & 0 \\ & & & \vdots \\ & & & 0 \\ & & & \\ 0 & \dots & 0 & 1/[v,v] \end{bmatrix}.$$

So the following observations are evident.

Corollary 4.8. *Given lines $l_1, \dots, l_5$ and vectors $a_1, \dots, a_5$ as above,* $([l_i, l_j])^{-1} = ([a_i, a_j])$.

Corollary 4.9. *Suppose we are given lines $l_1, \dots, l_5$ and vector v such that $\langle l_1, \dots, l_5 \rangle^{\perp} = \langle v \rangle$, and five two-vectors $a_1, \dots, a_5$ such that* $[\alpha_i, l_j] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

If all entries of $[\alpha_i, \alpha_j]$ have the same sign as $[v, v]$, then there does not exist a line l such that $[l, l_i] > 0$, for every $i = 1, \ldots, 5$.

The entries of $[l_i, l_j]^{-1}$ are the 4×4 cofactors of $([l_i, l_j])$ divided by $\det([l_i, l_j])$. This determinant always has the opposite sign of $[v, v]$ (by Sylvester's inertia theorem [17]). So the hypothesis of Corollary 4.9 is equivalent to: all 4×4 cofactors of $([l_i, l_j])$ are negative.

COROLLARY 4.10. *For $l_1, \ldots, l_5$, if $([l_i, l_j])$ has a positive diagonal 4×4 minor, then there exists a line l, such that $[l, l_i] > 0$ for $i = 1, \ldots, 5$.*

PROOF: If $\langle v \rangle = \langle l_1, \ldots, l_5 \rangle^{\perp}$, then a positive k^{th} diagonal minor gives an k^{th} diagonal entry of $([l_i, l_j])^{-1}$ opposite in sign to $[v, v]$. Therefore the a_i and v can be scaled so that $l = \sum_{i=1}^{5} \alpha_i a_i + \alpha_6 v$, with the $\alpha_i > 0$ $i = 1, \ldots, 6$ and thus $[l, l_i] > 0$ for every $i = 1, \ldots, 5$ and $[l, l] = 0$. ■

The k^{th} diagonal minor of $[l_i, l_j]$ is just the determinant of the form q over the rank-four space of the four lines obtained by deleting l_k from $l_1, \ldots l_5$. Recall that this determinant is a sign invariant (Theorem 4.2). The interpretation of this being positive is that any one of the four lines meets the regulus of the other three in two points. So for five lines, $l_1, \ldots, l_5$, a sufficient condition that there exists a line l such that $[l, l_i] > 0$ for every $i = 1, \ldots, 5$, is that one of the lines intersects the regulus of some other three in two points. In Example 4.6 none of the $\binom{5}{3}$ reguli generated by some three of the lines are intersected by either of the remaining two lines.

Sometimes in considering problems in line geometry in rank-four projective space we work with a basis. A question of which basis would be most useful arises. Since q is not positive definite there is no basis of lines orthonormal over q. Certain line arrangements have utility as a basis. The lines in the linear span of three skew lines form a regulus. We have previously asserted that three lines of a regulus and three

lines of the conjugate regulus span $\Lambda^2(R^4)$. The matrix of q over three lines of a regulus and three lines of the conjugate regulus make this evident. Suppose l_1, l_2, and l_3 are skew lines and l_4, l_5, and l_6 are lines of the regulus conjugate to $\langle l_1, l_2, l_3\rangle$. Assume $[l_1 l_2] = a, [l_1 l_3] = b, [l_2 l_3] = c, [l_4 l_5] = d, [l_4 l_6] = e$ and $[l_5 l_6] = f$. Then the matrix of q over $l_1, \ldots, l_6$ is

$$\begin{bmatrix} 0 & a & b & 0 & 0 & 0 \\ a & 0 & c & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & d & 0 & f \\ 0 & 0 & 0 & e & f & 0 \end{bmatrix},$$

which is nonsingular. The nonsingularity of the matrix of q over a set of two–vectors is sufficient to guarantee the independence of the two–vectors but is not necessary (e.g. five lines of a singular complex give a singular matrix).

LEMMA 4.11. *Let l_1, l_2 and l_3 be three skew lines and l_4, l_5 and l_6 three distinct lines of the regulus conjugate to $\langle l_1, l_2, l_3\rangle$. Then $sign([l_1, l_2][l_1, l_3][l_2, l_3]) = -sign([l_4, l_5][l_4, l_6][l_5, l_6])$. A regulus and its conjugate regulus have opposite orientations.*

PROOF: Sylvester's Inertia Thoerem [17] states that a quadratic space over a bilinear form q can be decomposed into $W_1 \oplus W_2 \oplus W_0$ where W_1 is positive definite, W_2 is negative definite, and q is identically 0 over W_0; and moreover the dimension of the W_i is invariant. Therefore the signature of the discriminant of q is a sign invariant. The discriminant of the form $[l_i, l_j]$ is negative (this can be checked with an arbitrary basis, a convenient choice is six lines that form a tetrahedron). For l_1, l_2 and l_3 skew and l_4, l_5 and l_6 distinct lines of the conjugate regulus the discriminant is

$$\begin{vmatrix} 0 & [l_1, l_2] & [l_1, l_3] & & & \\ [l_1, l_2] & 0 & [l_2, l_3] & & 0 & \\ [l_1, l_3] & [l_2, l_3] & 0 & & & \\ & & & 0 & [l_4, l_5] & [l_4, l_6] \\ & 0 & & [l_4, l_5] & 0 & [l_5, l_6] \\ & & & [l_4, l_6] & [l_5, l_6] & 0 \end{vmatrix}.$$

This determinant being negative implies $([l_1,l_2][l_1,l_3][l_2,l_3])$ and $([l_4,l_5][l_4,l_6][l_5,l_6])$ are opposite in sign. ∎

Returning to the previous considerations we make the following observation.

LEMMA 4.12. *If lines $l_1,\dots,l_6$ are such that segments of these six lines form the boundary of a tetrahedron, then there are at least 16 of the 2^6 sequences of six + or − signs for which no line l can be found such that $\{sign([l,l_1]),\dots,sign([l,l_6])$ gives the sequence.*

PROOF: Let $l_1,\dots,l_6$ be as labeled in the figure.

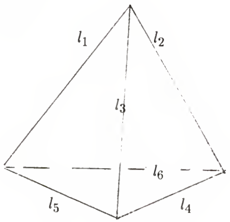

Figure 4.3

We may assume without loss of generality that the l_i are oriented and scaled so that $[l_1,l_4]=[l_2,l_5]=[l_3,l_6]=1$. The l_i form a basis of $\Lambda^2(R^4)$ (note that the discriminant of q over $l_1,\dots,l_6$ is non-zero). So an arbitary line $l=\sum_{i=1}^{6}\alpha_i l_i$. $[l,l]=0 \Longrightarrow \alpha_1\alpha_4+\alpha_2\alpha_5+\alpha_3\alpha_6=0$.

So $[l,l_1]=\alpha_4$, $[l,l_2]=\alpha_5$, $[l,l_3]=\alpha_6$, $[l,l_4]=\alpha_1$, $[l,l_5]=\alpha_2$, $[l.l_6]=\alpha_3$. The eight sequences where the pairs $\{\alpha_1,\alpha_4\}$, $\{\alpha_2,\alpha_5\}$, and $\{\alpha_3,\alpha_6\}$ agree in signs force the quadratic condition to fail. Likewise the eight sign sequences where the terms

of the pairs $\{\alpha_1, \alpha_4\}$, $\{\alpha_2, \alpha_5\}$ and $\{\alpha_3, \alpha_6\}$ have opposite signatures give α_i where $l = \sum_{i=1}^{6} \alpha_i l_i$ cannot be lines. ■

CHAPTER 5

SUMMARY AND CONCLUSION

The mid 1970s saw a rejuvenation of interest in invariant theory. With the formalization of the Cayley algebra there were promising avenues to attack classical problems of geometry and invariant theory. The Cayley algebra motivated the investigations that led to this dissertation.

The first observations featured in this research focus on linear invariant functions of extensors. We establish that multilinear invariant functions of extensors can be realized as dotted bracket polynomials. We then equate dotted bracket expressions to polynomials in tableaux. We establish combinatorial identities in tableaux which are identities in dotted bracket polynomials that preserve dottings among vector factors of extensors. We then apply the identities to establish a straightening algorithm that defines a canonical form for linear invariant functions of extensors. This canonical form, invariants in the standard basis, expedites applications that compare invariant functions of extensors. For example, in the Cayley factorization algorithm [21] there are iterative comparisons of invariant functions which appreciate the economy of the tableaux identities.

Several avenues of future research are suggested by these observations. One prominent problem is to extend the Cayley factorization procedure to nonlinear invariant functions. Huang's recent result [16] that in projective rank–four dotted brackets generate all invariant functions of extensors suggests the utility of tableaux application to consideration of nonmultilinear invariants. There are inherent difficulties in nonmultilinearity.

Tableaux lend themselves to investigations of rectifiability. Rectifiability often makes interpretation of invariants easier. Not all invariants are rectifiable, but there is a question of to what extent you can achieve a rectified basis. For example, it can be shown that in projective rank–four, the space of linear invariants in six two–extensors is spanned by the superbracket and the fifteen rectified bracket expressions. Problems of this type are essentially problems of identifying fundamental types of invariants.

Other investigations are motivated by problems suggested by the numerous applications of invariants. A variety of settings generate bracket polynomials or otherwise invite invariant theoretic analysis.

Chapter Four of this dissertation focuses on nonlinear invariants of two–extensors in projective rank–four space. We define the bracket of two lines (two–extensors) as a bilinear form on $\Lambda^2(R^4)$ which we denote $[l_1, l_2]$ for lines l_1 and l_2. We derive theorems about particular invariants and employ these to resolve the following problem about arrangements of lines:

> If we are given k lines $l_1, \ldots, l_k$, a line l' skew to these lines generates a sequence of signs, $\{\mathrm{sign}([l_1, l']), \ldots, \mathrm{sign}([l_k, l'])\}$. For an arbitrary k lines when is it possible to generate all 2^k sign sequences?

We establish that given four independent skew lines there are lines that generate each of the 16 sign sequences. We then provide a certain minimal example of five independent skew lines for which certain sign patterns cannot be established. We use invariants to analyze the relationship between these lines. Many problems of discrete structures that initially appear different can be translated to this type of problem.

This current wave of interest in invariant theory has produced many constructive devices for analyzing a variety of questions. It is evident that when or if this wave recedes invariant theory will be substantially richer in theory and application.

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BIOGRAPHICAL SKETCH

Timothy R. McMillan III was born in Fayetteville, North Carolina, in 1949. He grew up in Myrtle Beach, South Carolina, and graduated from Myrtle Beach High School in 1967. He attended Virginia Polytechnic Institute for one year. In 1976 he enrolled in the University of South Carolina. He received a B.S. in mathematics in May, 1979. In September, 1979 he enrolled in graduate school at the University of Florida. He received an M.S. in December, 1982. He taught intermittently at Santa Fe Community College, Gainesville, Florida. His Ph.D. in mathematics is expected in August, 1990.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Neil White, Chairman
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August 1990

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